Chapter 1

Bornology

1.1 Definitions

1.1.1

A bornology on a set $X$ is a family $B$ of subsets of $X$ satisfying the following axioms:

$B$ is covering of $X$
$B$ is hereditary under inclusion
$B$ is stable under finite union

$(X, B)$ is called a bornological set, and the elements of $B$ are called bounded subsets of $X$.

A base of a bornology $B$ is a subfamily $B_0$ of $B$, such that each element of $B$ is a subset of an element of $B_0$.

1.1.2

Let $E$ be a VS over $K$.
A bornology $B$ on $E$ is called a vector bornology on $E$ if:

$B$ is stable under vector addition
$B$ is stable under homothetic transformations
$B$ is stable under formation of circled hulls

$(E, B)$ is called a bornological vector space.

1.1.3

A vector bornology $B$ is called a convex vector bornology if it is stable under formation of convex hulls.

$(E, B)$ is called a convex bornological (vector) space.
1.1.4

A separated bornological vector space \((E,B)\) is a bornological VS, for which \(\{0\}\) is the only bounded subVS of \(E\).

1.2 Bounded Linear Maps

A map \(u\) between two bornological set \(X,Y\) is called a bounded map, if the image of each bounded subset of \(X\) is a bounded subset of \(Y\).

A bornology \(B_1\) on a set \(X\) is called a finer bornology than a bornology \(B_2\) on \(X\), if \(id : (E,B_1) \to (E,B_2)\) is bounded.

A bijection \(u\) between two bornological sets \(X,Y\) is called a bornological isomorphism if \(u\) and \(u^{-1}\) are bounded.

1.2.1

A bounded linear functional on a bornological VS \(E\) over \(K\) is a bounded linear map of \(E\) into \(K\), endowed with the bornology defined by the absolute value.

1.3 Fundamental Examples of Bornologies

Example 1:
Let \(K\) be a field with an absolute value. The family of subsets which are bounded in respect to the absolute value is a convex bornology on \(K\). It is called the canonical bornology of \(K\).

Example 2:
Let \(E\) be a VS over \(K\) and let \(p\) be a seminorm on \(E\). The collection of subsets \(A\) of \(E\) for which \(p(A)\) is bounded in \(K\) is called the canonical bornology of the seminormed space \((E,p)\). Note that this bornology is separated iff \(p\) is a norm.

Example 3:
Let \(\Gamma = \{p_i\}\) be a collection of seminorms on \(E\). Then the collection of subsets \(A\) of \(E\) for which \(p_i(A)\) is bounded in \(K\) is a bornology on \(E\). It is called the bornology defined by \(\Gamma\). It is separated iff \(\Gamma\) separates \(E\).

Example 4:
Let \(E\) be a topological VS. Then the collection of subsets \(A\) of \(E\) which are absorbed by each neighborhood of \(0\) is a vector bornology on \(E\). It is called the von Neumann bornology \(B\) of \(E\).

If \(E\) is locally convex, then so is \(B\).

Example 5:
Let \(E\) be a topological Hausdorff space. Then the family of relatively compact subsets of \(E\) is a vector
bornology on $E$, with the family of closed subsets of $E$ as a base. It is called the **compact bornology** of a topological space.

**Example 6:**
Let $E$ be a topological Hausdorff space. The family of subsets of compacts disks in $E$ is a convex bornology on $E$. It is called the **bornology of compact disks** of a topological space.

**Example 7:**
Let $E, F$ be topological VS. By $L(E, F)$ we denote the vector space of all continuous linear maps of $E$ into $F$.

A subset $H$ of $L(E, F)$ is called equicontinuous if:
For all neighborhoods $V$ of $0$, $V \in F, \quad H^{-1} := \bigcap_{u \in H} u^{-1}(V)$ is a neighborhood of 0 in $E$.

The family $K$ of equicontinuous subsets of $L(E, F)$ is a vector bornology on $L(E, F)$, and it is convex if $F$ is locally convex.

Since every element of $L(E, F)$ is continuous, $K$ covers $L(E, F)$ and is hereditary under inclusion and finite union.

Let $\mathfrak{V}$ be a base of circled neighborhoods in $F$. Then $\forall V \in \mathfrak{V} : \exists W \in \mathfrak{V}|W + W \subset V$

Let be $H_1, H_2 \in K$, then $H_1^{-1}(W)$ and $H_2^{-1}(W)$ are neighborhoods of zero in $E$.

$H_1^{-1}(W) \cap H_2^{-1}(W) \subset (H_1 + H_2)^{-1}(V)$, so $(H_1 + H_2)^{-1}(V)$ is an open neighborhood of 0 in $E$.

Since $V$ is an arbitrary element of a base of $F$, $K$ is stable under vector addition.

$K$ is also stable under homothetic transformations since
$(\lambda H)^{-1}(V) = \frac{1}{\lambda} H^{-1}(V)$

If $H_1$ is the circled hull of $H$, it is $H^{-1}(V) \subset H_1^{-1}(V)$, so $K$ is stable under formation of circled hulls.

Let $F$ be locally convex, and $\mathfrak{V}$ be a disked convex base of $F$.

Let be $x \in H^{-1}(V) \in \mathfrak{V}$. Then $\sum \lambda_i h_i(x) \in V$, since $V$ is convex. It follows that $H^{-1}(V) \subset (\Gamma H)^{-1}(V)$, so $K$ is convex.

**Example 8:**
let $X$ be a set, $\sigma$ a family of subsets of $X$, and $(F, B)$ a bornological set.

A family $C$ of maps of $X$ into $F$ is called $\sigma$-**bounded**, if $C(A)$ is bounded in $(F, B)$ for every $A \in \sigma$.

Let $H$ be a subset of all maps of $X$ into $F$. If all points in $H$ are $\sigma$-bounded, the $\sigma$-bounded subsets of $H$ define a bornology on $H$ called the $\sigma$-**bornology**.

If $(X, \sigma)$ is a bornological set, this is called the **natural bornology on** $H$.

A subset of $H$ which is bounded in respect to the natural bornology is called **equibounded**.
Chapter 2

Topology-Bornology: Internal Duality

2.1 Compatible Topologies and Bornologies

2.1.1
Let $E$ be a VS. Then let $B$ be a bornology on $E$ and let $J$ be a vector topology on $E$.

$B$ and $J$ are called compatible if $B$ is finer than the von Neumann bornology of $(E, J)$.

2.1.2
A subspace of a bornological VS $(E, B)$ is called a bornivorous subset if it absorbs every bounded set of $E$.

Let $E$ be a convex bornological space and let $V$ be the family of all bornivorous disks in $E$.

We are going to show that $V$ is a base for the finest locally convex topology $J$ on $E$ compatible with the bornology of $E$.

The members of $V$ are by definition absorbent, convex and circled. $V$ is clearly stable under homothetic transformation and finite intersections, so $V$ is the base of a locally convex topology on $E$. Every bounded subset of $E$ is bounded in the von Neumann bornology of $(E, J)$.

If $J'$ is a locally convex topology on $E$ which is compatible with $B$, it has a base of bornivorous disks and so $J$ is finer than $J'$.

The topology $J$ is called the locally convex topology associated with the bornology $B$ of $E$.

$E$, endowed with that topology is denoted by $\mathcal{T}E$.

2.1.3
Let $(E, J)$ be a locally convex space. Then by definition the von Neumann bornology of $(E, J)$ is the coarsest convex bornology on $E$ compatible with $J$. 

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2.1.4

The bornology of $\mathbf{BTE}$ is always coarser than the bornology of $E$. It is called the **weak bornology** of $E$.

**Proposition:**
A bornology on $E$ and its weak bornology are equal iff the bornology of $E$ is a von Neumann bornology of a locally convex space.

The necessity is obvious, but to prove the sufficiency, we first need to prove the following lemma:

**Lemma:**
For every locally convex space $F$, it is $\mathbf{BF} = \mathbf{BTBF}$

**Proof:**
Since $\text{id} : \mathbf{TB}F \rightarrow F$ is continuous, $\text{id} : \mathbf{BTBF} \rightarrow \mathbf{BF}$ is bounded.

Conversely, if $V$ is a bounded subset of $\mathbf{BF}$, it is absorbed by each neighborhood of zero of $\mathbf{TB}F$, and therefore is bounded in $\mathbf{BTBF}$

The lemma proofs the proposition.

A convex bornology on $E$ is called a **topological bornology** if $E = \mathbf{BTE}$, i.e. if it is the von Neumann bornology of a locally convex space.

2.1.5

Let $(E, J)$ be a locally convex space. Then the topology of $\mathbf{TBE}$ is always finer than $J$.

**Proposition:**
Let $E$ be a locally convex space. Then $E = \mathbf{TBE}$ iff the topology of $E$ is the locally convex topology associated with a convex bornology on $E$.

**Proof:**
The necessity is obvious. To prove the sufficiency, we first prove the following Lemma:

**Lemma:**
For each convex bornological space $F$, it is $\mathbf{TF} = \mathbf{TBTF}$

**Proof:**
Since $\text{id} : F \rightarrow \mathbf{BT}F$ is bounded, $\text{id} : \mathbf{TF} \rightarrow \mathbf{TBT}F$ is continuous.

Conversely, let $V$ be a neighborhood of $0$ in $\mathbf{TF}$, then it is a union of bornivorous disks. Since the bounded subsets in $\mathbf{BT}F$ are those sets absorbed by each neighborhood of zero in $\mathbf{TF}$, $V$ is a union of bornivorous discs in respect to $\mathbf{BT}F$ in is therefore open in $\mathbf{TBT}F$. 

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The Lemma proofs the proposition.

Let $E$ be a locally convex space. The topology of $E$ is called a **bornological topology**, if $E = \mathcal{T}E$.

**Proposition:**
Every metrizable locally convex topology $E$ is bornological.

**Proof:**
We have to show that $E = \mathcal{T}E$. Since $\mathcal{T}E$ is always finer than $E$, it suffices to show that $id : E \to \mathcal{T}E$ is continuous.

This equivalent to show that every bornivorous disk of $\mathcal{B}E$ is a neighborhood of zero in $E$.

Such a disk absorbs all sequences which converge to $0$, and therefore is a neighborhood in $E$, by the following Lemma:

**Lemma:**
In a metrizable topological VS $E$, every circled set which absorbs all sequences converging to $0$ is a neighborhood of $0$.

### 2.2 Characterisation of Bornological Topologies

#### 2.2.1

Let $E, F$ be locally convex spaces, and let $u$ be a linear map $u : E \to F$.

If $u$ is continuous, it is bounded in respect to $\mathcal{B}E, \mathcal{B}F$. The converse is not true in general.

#### 2.2.2

We will show that the locally convex topologies for which each bounded linear map into a LCS is continuous are exactly the bornological topologies.

**Proof:**
Let $E$ be a LCS.

Assume $E$ to be a bornological LCS and $u$ to be a linear map $u : E \to F$.

The for each disked neighborhood $V$ of zero in $F$, $u^{-1}(V)$ is a bornivorous disk in $E$, and therefore a neighborhood of zero in $E$, since $E = \mathcal{T}E$.

Suppose every bounded (i.r.t. the von Neumann bornology) linear map $u : E \to F$ is continuous. Since the identity $id : \mathcal{B}E \to \mathcal{B}\mathcal{T}E$ is bounded, and therefore $id : E \to \mathcal{T}E$ is continuous. This leads to the topological identity $\mathcal{T}E = E$. 

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