Quantization of geometrical structures in locally covariant field theory

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[1] based on the joint work with Romeo Brunetti and Klaus Fredenhagen

Outline of the talk

1. Introduction
   - Effective quantum gravity
   - Local covariance
2. BV formalism for gravity
   - Kinematical structure
   - Dynamics and symmetries
   - BV complex
3. Quantization
   - Deformation quantization
   - Applications

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- It is not clear what should be an observable.
- Need for "background independance".
- What replaces the classical spacetime structure in Planck scale?

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- Understand the small scale structure of spacetime: relation to noncommutative geometry.

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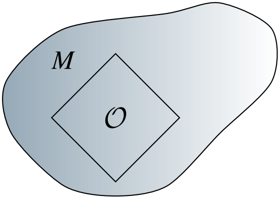

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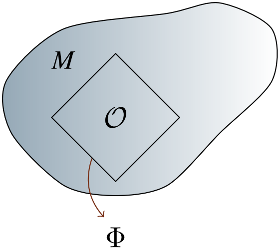

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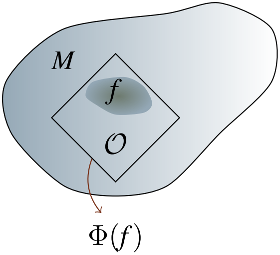

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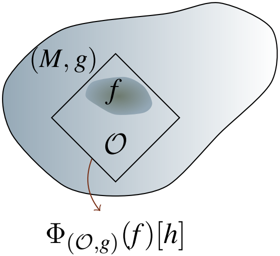

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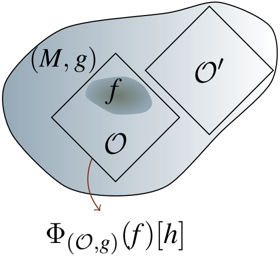
- Diffeomorphism transformation: move our experimental setup to a different region $\mathcal{O}'$.



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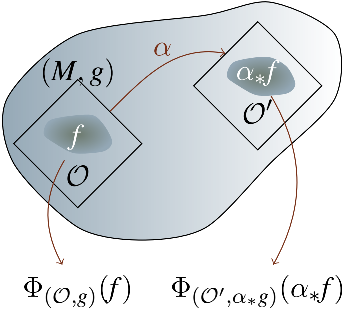

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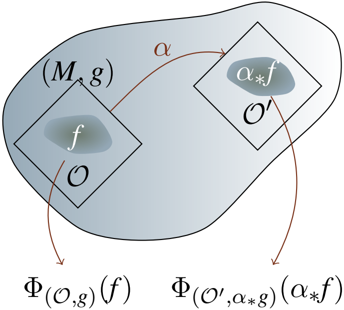

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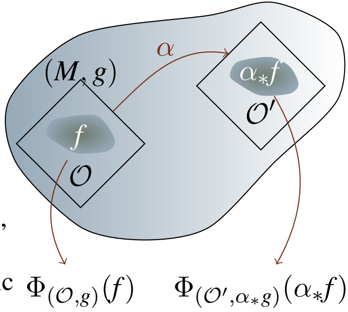

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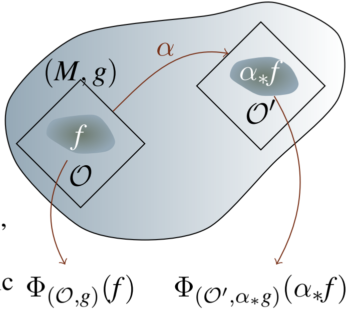
  - **Vec** with (small) topological vector spaces as objects and injective continuous homomorphisms of topological vector spaces as morphisms.



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- In a similar way we define a covariant functor $\mathfrak{E}_c : \mathbf{Loc} \to \mathbf{Vec}$ by setting $\mathfrak{E}\chi = \chi_*$, where:

$$\chi_* h \doteq \begin{cases} (\chi^{-1})^* h(x) & , \quad x \in \chi(M), \\ 0 & , \quad \text{else} \end{cases}$$

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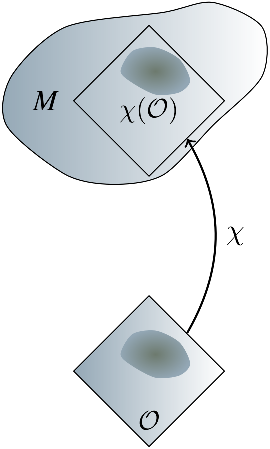
- $\mathfrak{V}(\mathcal{M}) \doteq$ the space of vector fields with above properties.
- $\mathfrak{V}$ becomes a (covariant) functor after setting: $\mathfrak{V}\chi(X) = \mathfrak{E}_c\chi \circ X \circ \mathfrak{E}\chi$.

Fields as natural transformations

- In the framework of locally covariant field theory [Brunetti-Fredenhagen-Verch 2003] fields are natural transformation between certain functors. For the sake of this talk let $\Phi \in \mathrm{Nat}(\mathfrak{D}, \mathfrak{F})$, where $\mathfrak{D}$ is the functor of test function spaces $\mathfrak{D}(\mathcal{M}) = \mathcal{C}^\infty_c(M)$ (one could substitute $\mathfrak{F}$ with a functor to the category of Poisson or C^* algebras).



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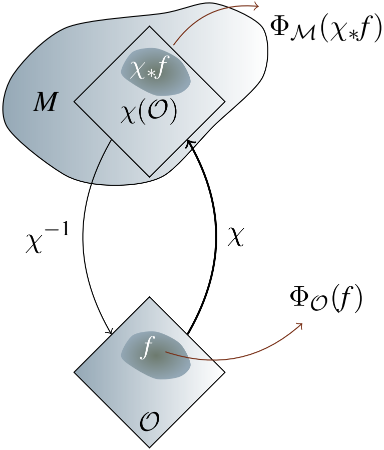

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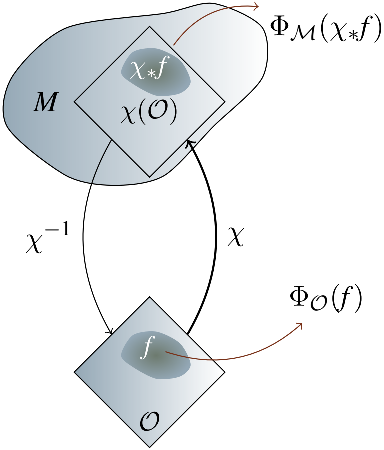
- The condition for Φ to be a natural transformation: $\Phi_\mathcal{O}(f)[\chi^* h] = \Phi_\mathcal{M}(\chi_* f)[h]$.
- In classical gravity we understand physical quantities not as pointwise objects but rather as something defined on all the spacetimes in a coherent way.



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- The dynamics is introduced by a generalized Lagrangian L which is a natural transformation between functors $\mathfrak{D}$ and $\mathfrak{F}_{\mathrm{loc}}$. The action $S(L)$ is an equivalence class of Lagrangians, where $L_1 \sim L_2$ if $\mathrm{supp}(L_{1,\mathcal{M}} - L_{2,\mathcal{M}})(f) \subset \mathrm{supp} df\ \forall f \in \mathfrak{D}(\mathcal{M})$.

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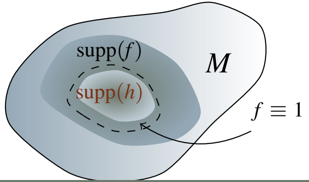
- The E-L derivative of $S(L)$ is a natural transformation $S' : \mathfrak{E} \to \mathfrak{E}'_c$ defined as $\left\langle S'_{\mathcal{M}}(h_0), h \right\rangle = \left\langle L_{\mathcal{M}}(f)^{(1)}(h_0), h \right\rangle$, where $f \equiv 1$ on $\mathrm{supp}h$. The field equation is: $S'_{\mathcal{M}}(h_0) = 0$. The space of solutions is denoted by $\mathfrak{E}_S(\mathcal{M})$.



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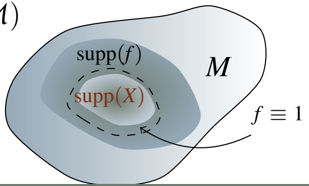
- A symmetry of S is a direction in $\mathfrak{E}(\mathcal{M})$ in which the action is constant, i.e. it is a vector field $X \in \mathfrak{V}(\mathcal{M})$ such that $\forall h_0 \in \mathfrak{E}(\mathcal{M})$: $0 = \langle S'_{\mathcal{M}}(h_0), X(h_0)\rangle$.



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- We can now define a Lie algebra $\mathcal{X}$, which provides us with a notion of transforming all the spacetimes in a coherent way.:

$$\mathcal{X} \doteq \prod_{\mathcal{M}\in \mathrm{Obj}(\mathbf{Loc})} \mathfrak{X}(\mathcal{M})$$

Diffeomorphism invariance

- Let $\vec{\xi} \in \mathcal{X}$ with all the components compactly supported and $\alpha_{\mathcal{M}} = \exp(\xi_{\mathcal{M}})$ a family of diffeomorphisms constructed via the exponential mapping. The action of diffeomorphisms on natural transformations is given by:
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- Example: $\int R[\tilde{g}] f \,\mathrm{d\,vol}_{(M,\tilde{g})}$ is invariant, but $\int R[\tilde{g}] f \,\mathrm{d\,vol}_{(M,g)}$ is not.

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- Idea: note that $\mathfrak{E}_S(\mathcal{M})$ locally can be seen critical manifold of the Lagrangian $L_{\mathcal{M}}(f) : \mathfrak{E}(\mathcal{M}) \to \mathbb{R}$ (zero locus of $S'_{\mathcal{M}}$).

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- $\mathfrak{F}_S(\mathcal{M}) = H_0\left(\bigwedge \mathfrak{V}(\mathcal{M}), S'_\mathcal{M}(.)\right)$.
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- To incorporate the gauge invariance we replace the original configuration space $\mathfrak{E}(\mathcal{M})$ with a graded manifold $\overline{\mathfrak{E}}(\mathcal{M}) \doteq \mathfrak{E}(\mathcal{M}) \oplus \mathfrak{X}(\mathcal{M})$ characterized by it's algebra of functions $\mathfrak{F}(\mathcal{M})\hat{\otimes}\bigwedge \mathfrak{X}'(\mathcal{M}) = \mathcal{C}^\infty_{\mathrm{ml}}\Big(\mathfrak{E}(\mathcal{M}), \bigwedge \mathfrak{V}(\mathcal{M})\Big)$.
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- Extending the algebra of functionals to the BV complex implies that the generalized fields should be also extended:

$$Fld = \bigoplus_{k=0}^{\infty} \mathrm{Nat}(\mathfrak{E}_c^k, \mathfrak{BV}) \,,$$

where $\mathfrak{E}_c^k$ be a functor from the category **Loc** to the product category $\mathbf{Vec}^k$, that assigns to a spacetime $\mathcal{M}$ a k-fold product of the test section spaces $\mathfrak{E}_c(\mathcal{M}) \times \ldots \times \mathfrak{E}_c(\mathcal{M})$.

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- The set *Fld* becomes a graded algebra if we set:

$$\begin{aligned}(\Phi\Psi)_{\mathcal{M}}(f_1, ..., f_{p+q}) &= \\ &= \frac{1}{p!q!} \sum_{\pi \in P_{p+q}} \Phi_{\mathcal{M}}(f_{\pi(1)}, ..., f_{\pi(p)}) \Psi_{\mathcal{M}}(f_{\pi(p+1)}, ..., f_{\pi(p+q)}) .\end{aligned}$$

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  $(s\Phi)_{\mathcal{M}}(f) = \{\Phi_{\mathcal{M}}(f), S + \gamma\} + \Phi_{\mathcal{M}}(\pounds_C f)$,
  where $C \in \mathfrak{X}(M)$ is the ghost and γ is the Chevalley-Eilenberg differential, which acts on *Fld* via infinitesimal diffeomorphism transformations along the ghost fields C. For $\Phi \in \mathrm{Nat}(\mathfrak{E}_c, \mathfrak{F})$:

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- For example: $\Phi_{(M,g)}(f)(h) = \int\limits_M R_{\mu\nu\alpha\beta}[\tilde{g}] R^{\mu\nu\alpha\beta}[\tilde{g}] f d\mathrm{vol}_{(M,\tilde{g})}$.

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- It holds: $H^0(s^\Psi, \alpha_\Psi(\mathit{Fld})) = H^0(s, \mathit{Fld}) = \mathit{Fld}_{\mathrm{inv}}$.

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- This Poisson structure can be naturally extended to a Poisson bracket $\{.,.\}_0$ on *Fld*.

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- The deformation quantization of $(\mathfrak{BV}_{\mu c}(\mathcal{M}), \{.,.\}_0^g)$ can be performed in the standard way, by introducing a $\star$-product:

$$(F \star_H G) \doteq m \circ \exp(i\hbar\Gamma_{\omega_H})(F \otimes G) \ ,$$

where $\Gamma_{\omega_H} \doteq \frac{1}{2}\int dx\, dy \omega_H(x,y)\frac{\delta}{\delta\varphi(x)} \otimes \frac{\delta}{\delta\varphi(y)}$ and $\omega_H = \frac{i}{2}\Delta_g + H$ is the Hadamard 2-point function (satisfies the linearized EOM's in both arguments and the μSC).

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- One can define the formal S-matrix as: $\mathcal{S}(V^g) \doteq e_{\mathcal{T}}^{V^g}$.
- Interacting fields are obtained from free ones in $\{Fld_{\mu c}[[\hbar, \lambda]], \star\}$ by the Bogoliubov formula:

$$(R_V(\Phi))_{\mathcal{M}}(f) \doteq \frac{d}{d\lambda}\Big|_{\lambda=0} \mathcal{S}(V^g)^{\star -1} \star \mathcal{S}(V^g + \lambda \Phi_{\mathcal{M}}(f)) \,.$$

Quantized geometrical structures

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- The BV construction can be applied the algebra of fields *Fld* and gives a homological interpretation to the notion of *gauge invariant physical quantities* in general relativity.
- The algebra $\mathit{Fld}_{\mu c}$ can be equipped with the noncommutative $\star$-product, which provides the deformation quantization of the free theory. The interaction is next introduced in the perturbative way and we obtain a notion of interacting quantum fields $R_V(\Phi)$, where Φ is a classical field constructed covariantly from the metric. For example: $\int R[\tilde{g}]f \,\mathrm{d}\,\mathrm{vol}_{(M,\tilde{g})}$.

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- The background independence would mean that the algebraic structure on $Fld_{\mu c}$ doesn't depend on the split into the free and interacting part. This can be obtained as a certain renormalization condition called perturbative agreement ([Hollands, Wald, 2004] for the scalar field). Work in progress.
- The physical interpretation of the theory is provided by constructing states on $\{Fld_{\mu c}[[\hbar, \lambda]], \star\}$. This problem is not entirely solved, since one needs to prove the existence of "gauge invariant" states on arbitrary $\mathcal{M}$ for the linearized theory. Up to now states can be explicitly given only on some special classes of spaetimes (for example ultrastatic).

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- The principle of *gravitational stability against localization of events* was proposed in [Doplicher, Fredenhagen, Roberts, CMP 95]. It states that a physical reason for Planck scale noncommutativity of spacetime is the fact that we cannot measure all the spacetime coordinates with arbitrary precision, because this would result in forming a trapped surface. This principle was used in [DFR 95] to derive the STUR in a simplified model.

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- In a recent work of [Doplicher, Morsella, Pinamonti 2012] this problem is studied for a model of scalar field coupled to gravity in the semiclassical approximation. We hope that using our notions of quantized metric and curvature one can move this one step further and include quantum gravity corrections.

Thank you for your attention