New notion of a renormalized time-ordered product in pAQFT

Katarzyna Rejzner[1]

II. Institute for Theoretical Physics, Hamburg University
University of Rome Tor Vergata

PM 2012, Madrid, 04.06.2012

[1] Based on the joint work with Klaus Fredenhagen.

Outline of the talk

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- A topological $*$- algebra is a topological vector space, which is a $*$- algebra and all the algebraic operations are continuous.

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  - Configuration space: space of smooth sections of some vector bundle $E \xrightarrow{\pi} M$ over M, for the scalar field: $\mathcal{E}(M) \equiv \mathcal{C}^\infty(M,\mathbb{R})$.
  - Action: a map $S_M : \mathcal{D}(M) \to \mathcal{C}^\infty(\mathcal{E}(M),\mathbb{R})$, where $\mathcal{D}(M) \equiv \mathcal{C}^\infty_c(M,\mathbb{R})$ are compactly supported smooth functions. An example action:
    $S_M(f)(\varphi) = \frac{1}{2}\int(\nabla_\mu\varphi\nabla^\mu\varphi - m^2\varphi^2)(x)f(x)d\mu(x)$, where $f \in \mathcal{D}(M)$ (cutoff), $\varphi \in \mathcal{E}(M)$, $\mu(x)$ is the volume form on M.

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  $$F^{(1)}(\varphi)(h) \doteq \lim_{t\to 0} \frac{1}{t}\left(F(\varphi + th) - F(\varphi)\right) \quad \text{(if exists)}$$

  F is called differentiable if $F^{(1)}(\varphi)(h)$ exists $\forall \varphi \in U$, $h \in \mathcal{E}(M)$. It is called continuously differentiable if it is differentiable on U and $F^{(1)} : U \times \mathcal{E}(M) \to \mathbb{R}, (\varphi, h) \mapsto F^{(1)}(\varphi)(h)$ is continuous.

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- The support of $F \in \mathcal{C}^\infty(\mathcal{E}(M),\mathbb{R})$ is defined as:

$$\begin{aligned}\operatorname{supp} F = \{x \in M | \forall \text{ neighbourhoods } U \text{ of } x\ \exists \varphi, \psi \in \mathcal{E}(M),\\ \operatorname{supp}\psi \subset U \text{ such that } F(\varphi+\psi) \neq F(\varphi)\}\,.\end{aligned}$$

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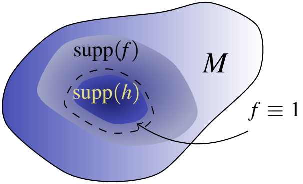
- F is local if it is of the form: $F(\varphi) = \int_M f(j_x(\varphi))\, d\mu(x)$, where f is a function on the jet bundle over M and $j_x(\varphi) = (\varphi(x), \partial\varphi(x), \ldots)$ is the jet of φ at the point x.

Equations of motion

- The Euler-Lagrange derivative of S_M is a map $S'_M : \mathcal{E}(M) \to \mathcal{D}'(M)$ defined as $\left\langle S'_M(\varphi), h \right\rangle = \left\langle L_M(f)^{(1)}(\varphi), h \right\rangle$, where $f \equiv 1$ on supp$h$.



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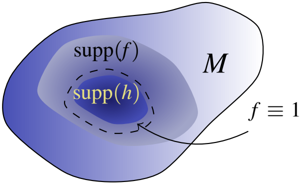

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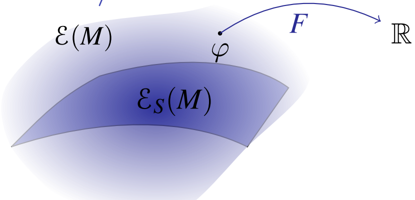

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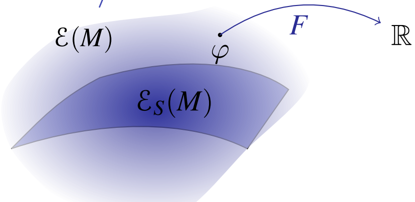

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- EOM determines a subspace of $\mathcal{E}(M)$ denoted by $\mathcal{E}_S(M)$ (on-shell configurations).
- We consider the space of smooth functionals on $\mathcal{E}(M)$, i.e. $\mathcal{C}^\infty(\mathcal{E}(M), \mathbb{R})$.

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- Let $\mathfrak{F}(M)_{\text{loc}}$ denote the space of local functionals and $\mathfrak{F}(M)$ the space of multilocal functionals (products of local ones).

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Definition

Let $u \in \mathcal{D}'(\Omega)$, *the wavefront set* $\mathrm{WF}(u)$ *is the complement in* $\Omega \times \mathbb{R}^n \setminus \{0\}$ *of the set of* $(x, \xi) \in \Omega \times \mathbb{R}^n \setminus \{0\}$ *such that there exist* $f \in \mathcal{D}(\Omega)$ *with* $f(x) = 1$ *and an open conic neighborhood* C *of* ξ, *with*

$$\sup_{\xi \in C}(1 + |\xi|)^N |\widehat{f \cdot u}(\xi)| < \infty \qquad \forall N \in \mathbb{N}_0 \,.$$

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- In a Lorentzian manifold there are some distinguished directions in TM and T^*M determined by the causal structure (i.e. causal, spacelike, timelike).
- We use these facts to define the algebra $\mathfrak{A}(M)$ by introducing a $\star$-product on a certain subspace of $\mathcal{C}^\infty(\mathcal{E}(M), \mathbb{R})$.

Free scalar field

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- Under some technical assumptions on M, P posses the retarded and advanced Green's functions Δ^R, Δ^A. They satisfy: $P \circ \Delta^{R/A} = \mathrm{id}_{\mathcal{D}(M)}$, $\Delta^{R/A} \circ (P|_{\mathcal{D}(M)}) = \mathrm{id}_{\mathcal{D}(M)}$ and

$$\mathrm{supp}(\Delta^R) \subset \{(x,y) \in M^2 | y \in (\overline{V}_-)_x\},$$
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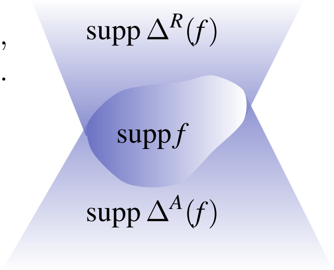

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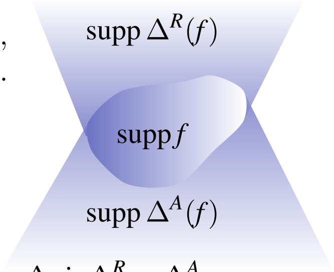

- Their difference is the causal propagator $\Delta \doteq \Delta^R - \Delta^A$.

Propagators and Green's functions

- $\mathrm{WF}(\Delta) = \{(x, k; x', -k') \in \dot{T}^*M^2 | (x, k) \sim (x', k')\}$, where $\sim$ means that there is a causal curve connecting x and x' and k' is the parallel transport of k along it.

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- We can always decompose Δ to positive and negative frequency parts: $i\Delta = \Delta_+ + \Delta_-$, i.e.:

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- We can also define the Feynman propagator:

$$\Delta_F = \frac{i}{2}(\Delta^A + \Delta^R) + \Delta_1\,.$$

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$$\mathrm{WF}(F^{(n)}(\varphi)) \subset \Xi_n, \quad \forall n \in \mathbb{N},\ \forall \varphi \in \mathfrak{E}(M)\,,$$

$$\Xi_n \doteq T^*M^n \setminus \{(x_1, \dots, x_n, k_1, \dots k_n) | k_i \in (\overline{V}_+)_{x_i} \cup (\overline{V}_-)_{x_i}, i = 1 \dots n\}.$$

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- We define the $\star$-product (deformation of the pointwise product):

$$(F \star G)(\varphi) \doteq \sum_{n=0}^{\infty} \left\langle F^{(n)}(\varphi), (\Delta_+)^{\otimes n} G^{(n)}(\varphi) \right\rangle \,,$$

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- The free QFT is defined as $\mathfrak{A}_0(M) \doteq (\mathfrak{F}_{\mathrm{mc}}(M), \star, *)$, where $F^*(\varphi) \doteq \overline{F(\varphi)}$.

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- Define the time-ordered product $\cdot_{\mathcal{T}}$ on $\mathcal{T}(\mathfrak{F}_{\mathrm{reg}}(M)[[\hbar]])$ by:

$$F \cdot_{\mathcal{T}} G \doteq \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G)$$

Interaction

- We now have an algebraic structure with two products $(\mathfrak{F}_{\text{reg}}(M)[[\hbar]], \star, \cdot_{\mathcal{T}})$, where $\star$ is non-commutative, $\cdot_{\mathcal{T}}$ is commutative and they are related by a causal relation:

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$$R_V(F) \doteq (e^{V}_{\mathcal{T}})^{\star -1} \star (e^{V}_{\mathcal{T}} \cdot_{\mathcal{T}} F)\,.$$
- Because of the WF set properties of Δ_F, the time-ordered product $\cdot_{\mathcal{T}}$ is not well defined on local, non-constant functionals, but the physical interaction is usually local!

Renormalization problem

- Renormalization problem: extend $\mathcal{S}(.)$ to local arguments. This is reduced to extending the n-fold time-ordered products, since we can define:

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- To extend $\mathcal{T}^n$ to arbitrary local functionals we use the causal approach of Epstein and Glaser (causal perturbation theory).

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3. **Causal factorization property**. If the supports of $F_1 \ldots F_i$ are later than the supports of $F_{i+1}, \ldots F_n$, then we have:

$$\mathcal{T}^n(F_1 \otimes \cdots \otimes F_n) = \mathcal{T}^i(F_1 \otimes \cdots \otimes F_i) \star \mathcal{T}^{n-i}(F_{i+1} \otimes \cdots \otimes F_n)\,.$$

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By the theorem of Epstein and Glaser we know that the extension exists, but is not unique. The theorem is proved inductively (in n) and at each step the problem is reduced to an extension of a real valued distribution.

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- The n-fold time-ordered product can then be written as

$$F_1 \cdot_{\mathcal{T}} \cdots \cdot_{\mathcal{T}} F_n = \sum_{\alpha \in \mathbb{N}^n} \left\langle \widetilde{S_\alpha}, \boldsymbol{\delta}^{\alpha}\left(F_1 \otimes \cdots \otimes F_n\right) \right\rangle$$

  where we denoted:

$$\widetilde{S_\alpha} \doteq \sum_{\Gamma \in \mathcal{G}_\alpha} \frac{\hbar^{|E(\Gamma)|}}{\mathrm{Sym}(\Gamma)} \widetilde{S_\Gamma}\,.$$

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- The distribution $\widetilde{S_\alpha}$ is a sum over $\mathcal{G}_\alpha$, the set of (non-tadpole) graphs with $n = \dim(\alpha)$ vertices and $\frac{|\alpha|}{2}$ lines such that there are α_i lines joining at vertex i and $\mathrm{Sym}(\Gamma) \in \mathbb{N}$ is the so called symmetry factor of the graph Γ.

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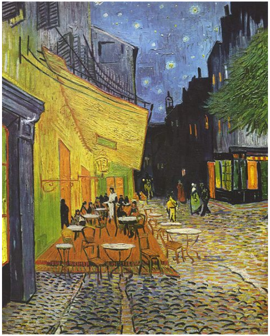
- Let $Z \in \mathcal{R}$, then we obtain a new S-matrix as $\widehat{\mathcal{S}} = \mathcal{S} \circ Z$ and all the S-matrices are obtained in this way.

Renormalized *n*-fold time-ordered products



- We have already maps $\mathcal{T}^n : \mathfrak{F}_{\mathrm{loc}}(M)^{\otimes n} \to \mathfrak{F}_{\mathrm{mc}}(M)[[\hbar]]$, but we can get even more! There exists a map $\beta : \mathfrak{F}(M) \to S^\bullet \mathfrak{F}^{(0)}_{\mathrm{loc}}(M)$, inverse to the pointwise multiplication.

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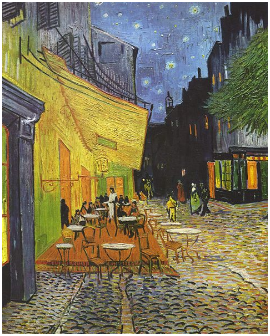

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- We define $\mathcal{T}_{\mathrm{r}} = (\oplus_n \mathcal{T}_{\mathrm{r}}^n) \circ \beta$ and propose an new notion of a time ordered product:

$$F \cdot_{\mathcal{T}_{\mathrm{r}}} G \doteq \mathcal{T}_{\mathrm{r}}(\mathcal{T}_{\mathrm{r}}^{-1} F \cdot \mathcal{T}_{\mathrm{r}}^{-1} G)\,,$$

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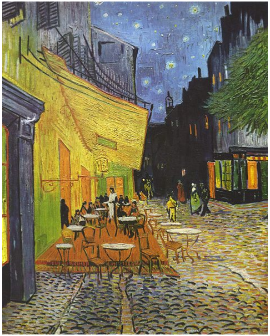

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- The renormalized QFT is a structure with two products $\mathfrak{A}(M) = (\mathcal{T}_{\mathrm{r}}(\mathfrak{F}(M)), \star, \cdot_{\mathcal{T}_{\mathrm{r}}})$.

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Thank you for your attention!