The Classical and the Quantum Master Equation in Locally Covariant Field Theory

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Outline of the talk

1. Local covariance
   - Kinematical structure
   - Equations of motion and symmetries
   - Antibracket and the CME
2. Quantization
   - pAQFT
   - QME and the quantum BV operator
   - Renormalized time-ordered products

This talk is based on:

- K. Fredenhagen, K. R., *Batalin-Vilkovisky formalism in perturbative algebraic quantum field theory*, [arXiv:math-ph/1110.5232],

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- K. R., *Batalin-Vilkovisky formalism in locally covariant field theory*, Ph.D. thesis.

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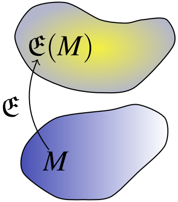
Kinematical structure

In our formulation, with a physical system we associate:

- The configurations space $\mathfrak{E}(M)$ of all fields of the theory. $\mathfrak{E}$ is a contravariant functor from **Loc** (spacetimes) to **Vec** (lcvs). For the scalar field $\mathfrak{E}(M) = \mathcal{C}^{\infty}(M)$.



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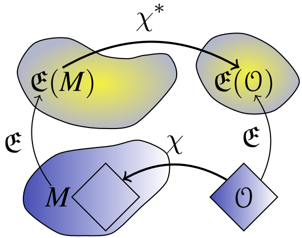

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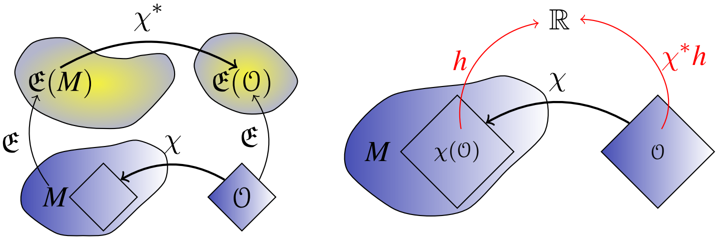

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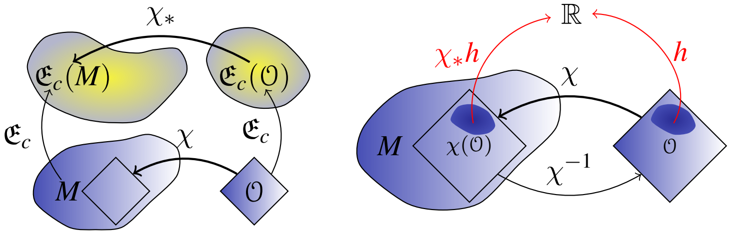

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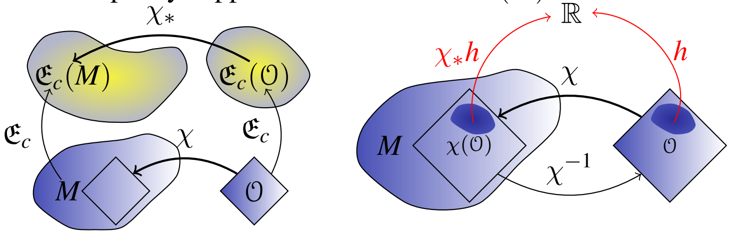
- $\mathfrak{D}$: **Loc** $\to$ **Vec** a covariant functor that assigns to M the space of compactly supported test functions $\mathfrak{D}(M)$.



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- The spacetime support of a function $F \in \mathcal{C}^\infty(\mathfrak{E}(M))$ is defined:

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- In this talk we restrict ourselves to products of local functionals, we denote this space by $\mathfrak{F}(M)$.

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- The space of vector fields with above properties is denoted by $\mathfrak{V}(M)$. $\mathfrak{V}$ becomes a (covariant) functor by setting: $\mathfrak{V}\chi(X) = \mathfrak{E}_c\chi \circ X \circ \mathfrak{E}\chi$,

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- The action $S(L)$ is an equivalence class of Lagrangians. We say that $L_1 \sim L_2$ if $\forall f \in \mathfrak{D}(M)$, $M \in \text{Obj}(\mathbf{Loc})$:

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- For example: $L_M(f) = \int_M \left(\frac{1}{2}\varphi^2 + \frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi\right) f \,\mathrm{dvol}_M$.

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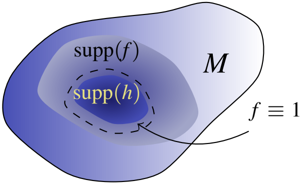
- The Euler-Lagrange derivative of S is defined by: $\left\langle S'_M(\varphi), h\right\rangle = \left\langle L_M(f)^{(1)}(\varphi), h\right\rangle, f \equiv 1$ on supp$h$. The field equation is: $S'_M(\varphi) = 0$. The space of solutions is denoted by $\mathfrak{E}_S(M) \subset \mathfrak{E}(M)$.



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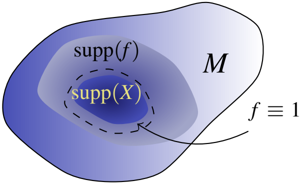

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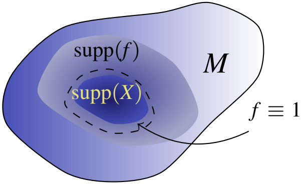
- In other words: a symmetry is a direction in $\mathfrak{E}(M)$ in which the action is constant. We denote the space of symmetries by $\mathfrak{s}(M)$.



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Invariants

- The space of symmetries $\mathfrak{s}(M)$ is a Lie subalgebra of $\mathfrak{V}(M)$ and has a natural action (as a Lie algebroid) on $\mathfrak{F}(M)$.
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BV complex

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  - $\{X, Y \wedge Z\} = \{X,Y\} \wedge Z + (-1)^{|Y|(|X|+1)} Y \wedge \{X,Z\}$.

BV differential in terms of the antibracket

- In contrast to the standard approach, we don't restrict neither to compact spacetimes nor to compactly supported configurations. Therefore derivation $\delta_S = \langle S'_M, . \rangle$ is not inner with respect to $\{., .\}$, but locally it can be written as:
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- The full BV differential is recovered from the sum of these two Lagrangians L and θ, i.e.:
  $sF = \{F, L^{\text{ext}}_M(f)\}, \quad f \equiv 1$ on supp$F$, $F \in \mathfrak{V}(M)$
  and the extended Lagrangian is defined as $L^{\text{ext}} \doteq L + \theta$.

Classical master equation

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- The nilpotency of s is equivalent to the extended classical master equation:

$$\{L^{\mathrm{ext}}, L^{\mathrm{ext}}\} \sim 0\,.$$

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- The time-ordering operator $\mathcal{T}$ is defined as: $\mathcal{T}(F) \doteq e^{i\hbar\Gamma_{\Delta_D}}(F)$, where $\Gamma_{\Delta_D} = \int dxdy \Delta_D(x,y) \frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)}$ and $\Delta_D = \frac{1}{2}(\Delta_R + \Delta_A)$ is the Dirac propagator.

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- Define the time-ordered product $\cdot_\mathcal{T}$ on $\mathcal{T}(\mathfrak{F}_{\mathrm{reg}}(M)[[\hbar]])$ by:

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Theorem (K. Fredenhagen, K.R. 2011)

The renormalized time-ordered product $\cdot_{\mathcal{T}_\mathrm{r}}$ is an associative product on $\mathcal{T}_\mathrm{r}(\mathfrak{F}(\mathbb{M}))$ given by

$$F \cdot_{\mathcal{T}_\mathrm{r}} G \doteq \mathcal{T}_\mathrm{r}(\mathcal{T}_\mathrm{r}^{-1} F \cdot \mathcal{T}_\mathrm{r}^{-1} G) \,,$$

with $\mathcal{T}_\mathrm{r} = (\oplus_n \mathcal{T}_n) \circ \beta$, where $\beta : \mathfrak{F}(\mathbb{M}) \to S^\bullet \mathfrak{F}_{\mathrm{loc}}^{(0)}(\mathbb{M})$ is the inverse of multiplication m.

Renormalized QME and the quantum BV operator

- Since $\cdot_{\mathcal{T}_r}$ is an associative, commutative product, we can use it in place of $\cdot_{\mathcal{T}}$ and define the renormalized QME and the quantum BV operator as:

$$0 = \{e_{\mathcal{T}_r}^{iV/\hbar}, S\}_\star\,,$$

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- These formulas get even simpler if we use the anomalous Master Ward Identity ([Brenecke-Dütsch 08, Hollands 08]). We obtained:

$$0 = \frac{1}{2}\{V+S, V+S\}_{\mathcal{T}_r} - \triangle_V(V)\,,$$
$$\hat{s}X = \{X, V+S\}_{\mathcal{T}_r} - \triangle_V(X)\,.$$

where $\triangle_V(X)$ is the anomaly. It is local and of order $\mathcal{O}(\hbar)$.

Renormalized QME and the quantum BV operator

- Since $\cdot_{\mathcal{T}_r}$ is an associative, commutative product, we can use it in place of $\cdot_{\mathcal{T}}$ and define the renormalized QME and the quantum BV operator as:

$$0 = \{e_{\mathcal{T}_r}^{iV/\hbar}, S\}_\star \,,$$
$$\hat{s}(X) = e_{\mathcal{T}_r}^{-iV/\hbar} \cdot_{\mathcal{T}_r} \left(\{e_{\mathcal{T}_r}^{iV/\hbar} \cdot_{\mathcal{T}_r} X, S\}_\star \right) \,.$$

- These formulas get even simpler if we use the anomalous Master Ward Identity ([Brenecke-Dütsch 08, Hollands 08]). We obtained:

$$0 = \frac{1}{2}\{V+S, V+S\}_{\mathcal{T}_r} - \triangle_V(V) \,,$$
$$\hat{s}X = \{X, V+S\}_{\mathcal{T}_r} - \triangle_V(X) \,.$$

where $\triangle_V(X)$ is the anomaly. It is local and of order $\mathcal{O}(\hbar)$.

- By using the renormalized time ordered product $\cdot_{\mathcal{T}}$ we obtained in place of $\triangle$, the interaction-dependent operator $\triangle_V$.

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- The quantum BV operator is defined as

$$\hat{s}(X) = e_{\mathcal{T}_r}^{-iS_{1M}(f_1)/\hbar} \cdot_{\mathcal{T}_r} \left(\{ e_{\mathcal{T}_r}^{iS_{1M}(f_1)/\hbar} \cdot_{\mathcal{T}_r} X, S_{0M}(f) \}_\star \right) \,,$$

where $\operatorname{supp} X \subset \mathcal{O}$ and $f, f_1 \equiv 1$ on $\mathcal{O}$.

Renormalization group action

Proposition (K. Fredenhagen, K.R. 2011)

Let L_1 be a natural Lagrangian that solves the QME for the renormalized time-ordered product $\mathcal{T}_\mathrm{r}$. Let $Z \in \mathcal{R}$ be the element of the renormalization group, which transforms between the S-matrices corresponding to $\mathcal{T}_\mathrm{r}$ and $\mathcal{T}_\mathrm{r}'$, i.e. $e_{\mathcal{T}_\mathrm{r}}^{L_{1M}(f)} = e_{\mathcal{T}_\mathrm{r}'}^{Z(L_{1M}(f))}$. Then $Z(L_1)$ solves the QME corresponding to $\mathcal{T}_\mathrm{r}'$.

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For the quantum BV operator we have a relation:

$$\hat{s}_{Z(S_1)} \circ Z^{(1)}(S_1) = Z^{(1)}(S_1) \circ \hat{s}'_{S_1} \, .$$

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- The renormalized QME and the quantum BV operator are defined in a natural way and don't suffer from divergent terms,
- We generalized these structures to the level of natural Lagrangians and showed that they transform correctly under the action of the renormalization group.

Thank you for your attention!