Functional analytic aspects of renormalization on Lorentzian manifolds

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Outline of the talk

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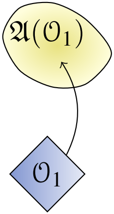

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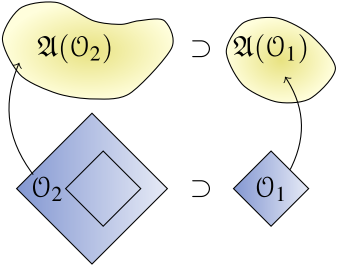
- The physical notion of subsystems is realized by the condition of isotony, i.e.: $\mathcal{O}_2 \supset \mathcal{O}_1 \Rightarrow \mathfrak{A}(\mathcal{O}_2) \supset \mathfrak{A}(\mathcal{O}_1)$. We obtain a net of algebras.



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- Here, a topological $*$- algebra is a topological vector space, which is a $*$- algebra and the product is sequentially continuous.

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  - Configuration space: space of smooth sections of some vector bundle $E \xrightarrow{\pi} M$ over M, for the scalar field: $\mathcal{E}(M) \equiv \mathcal{C}^\infty(M,\mathbb{R})$.
  - Action: a map $S_M : \mathcal{D}(M) \to \mathcal{C}^\infty(\mathcal{E}(M),\mathbb{R})$, where $\mathcal{D}(M) \equiv \mathcal{C}^\infty_c(M,\mathbb{R})$ are compactly supported smooth functions. An example action:
    $S_M(f)(\varphi) = \frac{1}{2}\int(\nabla_\mu\varphi\nabla^\mu\varphi - m^2\varphi^2)(x)f(x)d\mu(x)$, where $f \in \mathcal{D}(M)$ (cutoff), $\varphi \in \mathcal{E}(M)$, $\mu(x)$ is the volume form on M.

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  $$F^{(1)}(\varphi)(h) \doteq \lim_{t \to 0} \frac{1}{t} \left(F(\varphi + th) - F(\varphi) \right) \quad \text{(if exists)}$$

  F is called differentiable if $F^{(1)}(\varphi)(h)$ exists $\forall \varphi \in U$, $h \in \mathcal{E}(M)$. It is called continuously differentiable (in the sense of Bastiani) if it is differentiable on U and $F^{(1)} : U \times \mathcal{E}(M) \to \mathbb{R}, (\varphi, h) \mapsto F^{(1)}(\varphi)(h)$ is continuous.

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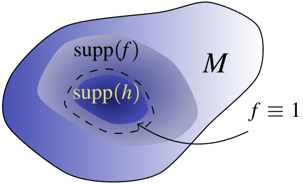
- The support of $F \in \mathcal{C}^\infty(\mathcal{E}(M), \mathbb{R})$ is defined as:

$$\begin{aligned}\operatorname{supp} F = \{x \in M | &\forall \text{ neighbourhoods } U \text{ of } x\ \exists \varphi, \psi \in \mathcal{E}(M),\\ &\operatorname{supp}\psi \subset U \text{ such that } F(\varphi + \psi) \neq F(\varphi)\}\,.\end{aligned}$$

Equations of motion

- The Euler-Lagrange derivative of S_M is a map $S'_M : \mathcal{E}(M) \to \mathcal{D}'(M)$ defined as $\langle S'_M(\varphi), h\rangle = \left\langle L_M(f)^{(1)}(\varphi), h\right\rangle$, where $f \equiv 1$ on supp$h$.



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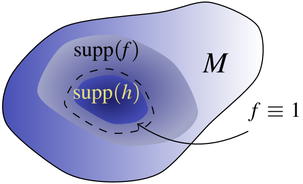

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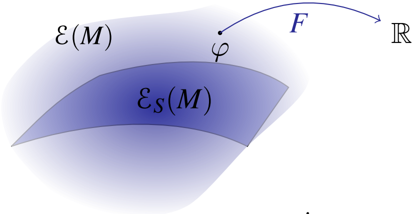

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- EOM determines a subspace of $\mathcal{E}(M)$ denoted by $\mathcal{E}_S(M)$ (on-shell configurations).

Regularity conditions

- F is local if it is of the form: $F(\varphi) = \int_M f(j_x(\varphi))\, d\mu(x)$, where f is a function on the jet bundle over M, $j_x(\varphi)$ is the jet of φ at x.

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Definition

Let $u \in \mathcal{D}'(\Omega)$, the wavefront set $\mathrm{WF}(u)$ *is the complement in $\Omega \times \mathbb{R}^n \setminus \{0\}$ of the set of $(x, \xi) \in \Omega \times \mathbb{R}^n \setminus \{0\}$ such that there exist $f \in \mathcal{D}(\Omega)$ with $f(x) = 1$ and an open conic neighborhood C of ξ, with*

$$\sup_{\xi \in C} (1 + |\xi|)^N |\widehat{f \cdot u}(\xi)| < \infty \qquad \forall N \in \mathbb{N}_0 \,.$$

Abstractn notion of locality

Definition

A functional $F \in \mathcal{C}^\infty(\mathcal{E}(M), \mathbb{R})$ *is called additive if*

$$F(\varphi_1 + \varphi_2 + \varphi_3) = F(\varphi_1 + \varphi_2) + F(\varphi_2 + \varphi_3) - F(\varphi_2)\,,$$

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2. F *is local.*

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- In a Lorentzian manifold there are some distinguished directions in TM and T^*M determined by the causal structure (i.e. causal, spacelike, timelike).
- Green's functions of a normally hyperbolic operator have some particular singularities structure.
- We use these facts to define the algebra $\mathfrak{A}(M)$ by introducing a $\star$-product on a certain subspace of $\mathcal{C}^\infty(\mathcal{E}(M),\mathbb{R})$.

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$$\mathrm{supp}(\Delta^R) \subset \{(x,y) \in M^2 | y \in (\overline{V}_-)_x\}\,,$$
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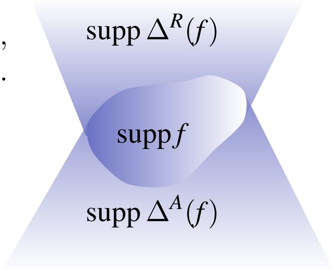

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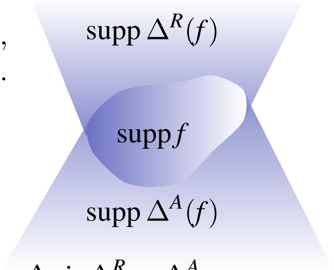

- Their difference is the causal propagator $\Delta \doteq \Delta^R - \Delta^A$.

Propagators and Green's functions

- $\mathrm{WF}(\Delta) = \{(x, k; x', -k') \in \dot{T}^*M^2 | (x, k) \sim (x', k')\}$, where $\sim$ means that there is a causal curve connecting x and x', and k' is the parallel transport of k along it.

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- We can always decompose Δ to positive and negative frequency parts: $\frac{i}{2}\Delta=\Delta_+-H$, i.e.:

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- We can also define the Feynman propagator:

$$\Delta_F = \frac{i}{2}(\Delta^A + \Delta^R) + H \,.$$

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$$\mathrm{WF}(F^{(n)}(\varphi)) \subset \Xi_n, \quad \forall n \in \mathbb{N},\ \forall \varphi \in \mathcal{E}(M)\,,$$

$$\Xi_n \doteq T^*M^n \setminus \{(x_1,\dots,x_n,k_1,\dots k_n) | k_i \in (\overline{V}_+)_{x_i} \cup (\overline{V}_-)_{x_i}, i = 1\dots n\}.$$

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- Define the $\star$-product (deformation of the pointwise product):

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- Introduce on $\mathfrak{F}_{\mu c}(M)$ a topology τ that controls the WF sets of the derivatives of functionals.

Net of $*$-algebras

Definition

The free QFT is defined by assigning to $\mathcal{O} \subset M$

$$\mathfrak{A}_0(\mathcal{O}) \doteq (\mathfrak{F}_{\mu c}(\mathcal{O}), \tau, \star, *)\,,$$

where $F^*(\varphi) \doteq \overline{F(\varphi)}$ and $\mathfrak{F}_{\mu c}(\mathcal{O})$ is the space of microcausal functionals supported in $\mathcal{O}$.

Time-ordered product

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$$\mathcal{T}F(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(2n)}(\varphi), \left(\tfrac{1}{2}\Delta_F\right)^{\otimes n} \right\rangle \ ,$$

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- Define the time-ordered product $\cdot_{\mathcal{T}}$ on $\mathfrak{F}_{\mathrm{reg}}(M)[[\hbar]]$ by:

$$F \cdot_{\mathcal{T}} G \doteq \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G)$$

Interaction

- We now have an algebraic structure with two products $(\mathfrak{F}_{\mathrm{reg}}(M)[[\hbar]], \star, \cdot_{\mathcal{T}})$, where $\star$ is non-commutative, $\cdot_{\mathcal{T}}$ is commutative and they are related by a causal relation:

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- Interaction is a functional $V \in \mathfrak{F}_{\mathrm{reg}}(M)$. Using the commutative product $\cdot_{\mathcal{T}}$ we define the S-matrix:
$$\mathcal{S}(V) \doteq e_{\mathcal{T}}^{V} = \mathcal{T}(e^{\mathcal{T}^{-1}V})\,.$$

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- Because of the WF set properties of Δ_F, the time-ordered product $\cdot_{\mathcal{T}}$ is not well defined on local, non-constant functionals, but the physical interaction is usually local!

Renormalization problem

- Renormalization problem: extend $\mathcal{S}(.)$ to local arguments. This is reduced to extending the n-fold time-ordered products, since we can define:

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- To extend $\mathcal{T}^n$ to arbitrary local functionals we use the causal approach of Epstein and Glaser (causal perturbation theory).

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By the theorem of Epstein and Glaser we know that the extension exists, but is not unique. The theorem is proved inductively (in n) and at each step the problem is reduced to an extension of a real valued distribution.

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  - distribution: $\widetilde{S_\alpha} \doteq \sum\limits_{\Gamma \in \mathcal{G}_\alpha} \frac{\hbar^{|E(\Gamma)|}}{\mathrm{Sym}(\Gamma)} \widetilde{S_\Gamma}$, where the sum is taken over $\mathcal{G}_\alpha$, the set of (non-tadpole) graphs with $n = \dim(\alpha)$ vertices and $\frac{|\alpha|}{2}$ lines such that there are α_i lines joining at vertex i and $\mathrm{Sym}(\Gamma) \in \mathbb{N}$ is the so called symmetry factor of the graph Γ.

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- Each $\widetilde{S_\Gamma} \doteq \prod_{e \in E(\Gamma)} \Delta_F(x_{e,i}, i \in \partial e)$ is a well defined distribution on $\mathcal{D}'((\mathbb{M}^2 \setminus \mathrm{Diag})^{|E(\Gamma)|})$ (Diag denotes the diagonal).

Causal perturbation theory

- The n-fold time-ordered product can then be written as

$$F_1 \cdot_{\mathcal{T}} \cdots \cdot_{\mathcal{T}} F_n = \sum_{\alpha\in\mathbb{N}^n} \left\langle \widetilde{S_\alpha}, \boldsymbol{\delta}^\alpha \left(F_1 \otimes \cdots \otimes F_n\right)\right\rangle$$

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- Functional derivatives of a local functional have the form

$$F^{(l)}[\varphi](x_1, \ldots, x_l) = \int dz \sum_j f_j[\varphi](z) p_j(\partial_{x_1}, \ldots, \partial_{x_l}) \prod_{i=1}^{l} \delta(z - x_i)$$

with polynomials p_j and φ-dependent test functions $f_j[\varphi]$.

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- Note that functional derivatives $\frac{\delta}{\delta\varphi(x_{e,v})}$ are associated to vertices v of the graph (see picture), and we get one derivative for each edge e adjacent at v, so the variables "x" in the formula above are also numbered by $v \in V(\Gamma)$ and $e \in E(\Gamma)$.

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- We can move the partial derivatives $\partial_{x_{e,v}}$ by formal partial integration to the distribution $\widetilde{S}_\Gamma$.

Extensions of distributions

- Next we integrate over the delta distributions, which amounts to the pullback with respect to $\rho_\Gamma : \mathbb{M}^{|V(\Gamma)|} \to \mathbb{M}^{2|E(\Gamma)|}$,

$$(\rho_\Gamma(z))_{e,v} = z_v \quad \text{if } v \in \partial e\,.$$

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- Let p be a polynomial in the partial derivatives $\partial_{x_{e,v}}$, $v \in \partial e$. The pullback ρ_Γ^* of $p\widetilde{S}_\Gamma$ is well defined on $\mathbb{M}^{|V(\Gamma)|}\setminus$DIAG, where DIAG is the large diagonal:

$$\text{DIAG} = \left\{ z \in \mathbb{M}^{|V(\Gamma)|} \,|\, \exists v, w \in V(\Gamma), v \neq w : \; z_v = z_w \right\}\,.$$

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- The problem of renormalization now amounts to finding the extensions of $\rho_\Gamma^* p\widetilde{S}_\Gamma$ to everywhere defined distributions $S_{\Gamma,p} \in \mathcal{D}'(\mathbb{M}^{|V(\Gamma)|})$.

Extensions of distributions

- The construction is inductive. We assume that graphs with $k < n$ vertices are renormalized (corresponding extensions of distributions are constructed), so the problem of renormalizing graphs with n vertices reduces to extending distributions defined everywhere outside the origin.

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- The existence and uniqueness of extensions $\dot{t}$ can be answered in terms of Steinmann's scaling degree of t,

$$\mathrm{sd}(u) := \inf\{\omega \in \mathbb{R} \mid \lim_{\rho\downarrow 0} \rho^{\omega}\, u(\rho x) = 0\},$$
$$u \in \mathcal{D}'(\mathbb{R}^n) \quad \text{or} \quad u \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})\,.$$

Extensions of distributions

Theorem (Steinmann 71, Brunetti Fredenhagen 2000)

For $\lambda \in \mathbb{R}$ let

$$\mathcal{D}_\lambda(\mathbb{R}^n) := \{f \in \mathcal{D}(\mathbb{R}^n) \,|\, (\partial^\alpha f)(0) = 0 \ \ \forall |\alpha| \le \lambda\} \tag{1}$$

(in particular $\mathcal{D}_\lambda(\mathbb{R}^n) = \mathcal{D}(\mathbb{R}^n)$ if $\lambda < 0$) and let $\mathcal{D}'_\lambda(\mathbb{R}^n)$ be the corresponding space of distributions. A distribution $t \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ with scaling degree $\mathrm{sd}(t)$ has a unique extension $\bar{t} \in \mathcal{D}'_\lambda(\mathbb{R}^n)$, $\lambda = \mathrm{sd}(t) - n$, which satisfies the condition $\mathrm{sd}(\bar{t}) = \mathrm{sd}(t)$.

An extension to a distribution on the full space $\mathcal{D}(\mathbb{M}^{n-1})$ can be therefore defined by a choice of the projection:

$$W : \mathcal{D}(\mathbb{M}^{n-1}) \to \mathcal{D}_\lambda(\mathbb{M}^{n-1})\,.$$

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  4. They don't depend explicitly on the field configuration φ.

Commutative product

- We have already constructed maps
  $\mathcal{T}^n : \mathfrak{F}_{\mathrm{loc}}(M)^{\otimes n} \to \mathfrak{F}_{\mu c}(M)[[\hbar]]$,
  but we can get even more! There exists a map
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- We define $\mathcal{T}_{\mathrm{r}} = (\oplus_n \mathcal{T}_{\mathrm{r}}^n) \circ \beta$ and set:

$$F \cdot_{\mathcal{T}_{\mathrm{r}}} G \doteq \mathcal{T}_{\mathrm{r}}(\mathcal{T}_{\mathrm{r}}^{-1} F \cdot \mathcal{T}_{\mathrm{r}}^{-1} G)\,,$$

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  which is an associative, commutative product on $\mathcal{T}_{\mathrm{r}}(\mathfrak{F}(M))$.
- The renormalized QFT is a structure with two products $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}) = (\mathcal{T}_{\mathrm{r}}(\mathfrak{F}(\mathcal{O})), \tau, \star, \cdot_{\mathcal{T}_{\mathrm{r}}})$.

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- Construction of QFT models on curved spacetimes can be put on solid mathematical grounds using the functional approach.
- The basic structure is a net of non-commutative topological $*$-algebras with the additional commutative product.
- Analytic tools involve calculus on locally convex topological vector space and methods of microlocal analysis.

Thank you for your attention!