Batalin-Vilkovisky formalism and General Relativity

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Outline of the talk

1. Preliminaries
   - Kinematical structure
   - Smooth calculus in lcvs
   - Equations of motion and symmetries

2. Gravity
   - Action and symmetries
   - BV complex on natural transformations

- Very successful in perturbative quantum field theory

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- Needs more fundamental structural understanding

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- R. Brunetti, K. Fredenhagen *Towards a Background Independent Formulation of Perturbative Quantum Gravity*, [arXiv:gr-qc/0603079v3].

Categories

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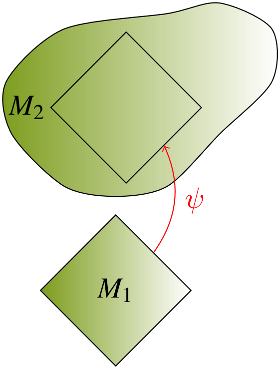

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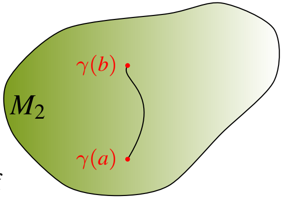

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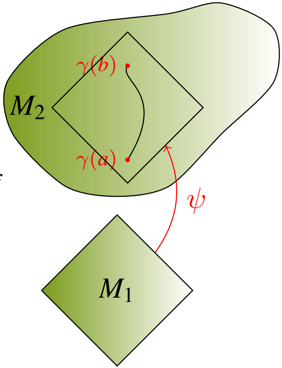

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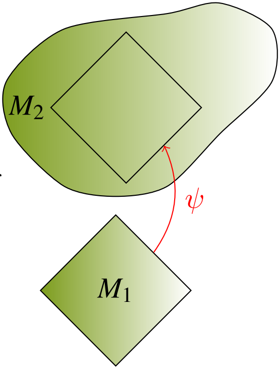

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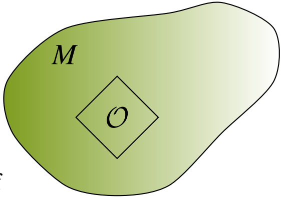

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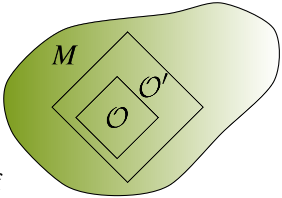

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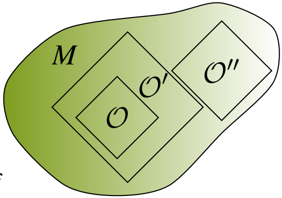

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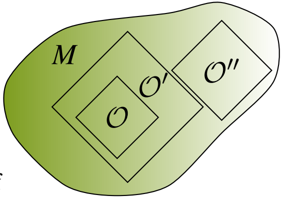

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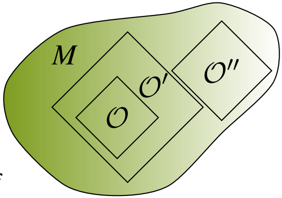

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Statement of the problem

In our formulation with a physical system we associate:

- The configurations space $\mathfrak{E}(M)$ of all fields of the theory. $\mathfrak{E}$ is a contravariant functor from **Loc** (spacetimes) to **Vec** (lcvs). For the scalar field: $\mathfrak{E}(M) = \mathcal{C}^\infty(M)$.

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- The space of smooth, compactly supported functionals on $\mathfrak{E}(M)$. This also defines a covariant functor $\mathfrak{F}$: **Loc** $\rightarrow$ **Vec** (+ regularity conditions: local, microcausal, ...).

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- F is local if it is of the form: $F(\phi) = \int_M f(j_x(\phi))\, d\mu(x)\,,$ where f is a function on the jet bundle over M and $j_x(\phi)$ is the jet of ϕ at the point x. $\mathfrak{F}_{\mathrm{loc}}$ is a subfunctor of $\mathfrak{F}$.

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- In this talk we restrict ourselves to *multilocal functionals*, which are defined as finite sums of finite products of local functionals.

Dynamics

- The dynamics is introduced by a generalized Lagrangian L which is a natural transformation between functors $\mathfrak{D}$ and $\mathfrak{F}_{\mathrm{loc}}$, s.t.: $\mathrm{supp}(L_M(f)) \subseteq \mathrm{supp}(f)$, and $L_M(\bullet)$ is additive in f. The action $S(L)$ is an equivalence class of Lagrangians. We say that $L_1 \sim L_2$ if $\forall f \in \mathfrak{D}(M)$, $M \in \mathbf{Loc}$:

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- For example: $L_M(f) = \int_M \left(\frac{1}{2}\phi^2 + \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi\right) f\, d\mathrm{vol}_M$.

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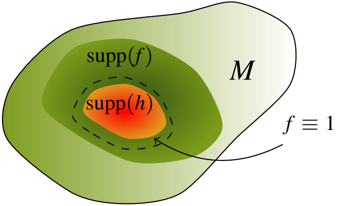
- We consider only the multilocal (products of local vector fields and local functionals) vector fields with compact support.
- The space of vector fields with above properties is denoted by $\mathfrak{V}(M)$. $\mathfrak{V}$ becomes a (covariant) functor by setting: $\mathfrak{V}\chi(X) = \mathfrak{E}_c\chi \circ X \circ \mathfrak{E}\chi$.

Equations of motion and symmetries

- The Euler-Lagrange derivative of S is a natural transformation $S' : \mathfrak{E} \to \mathfrak{D}'$ defined by: $\left\langle S'_M(\varphi), h \right\rangle = \left\langle L_M(f)^{(1)}(\varphi), h \right\rangle$ with $f \equiv 1$ on supp$h$. The field equation is: $S'_M(\varphi) = 0$.



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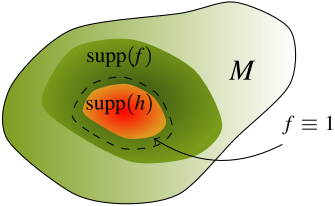

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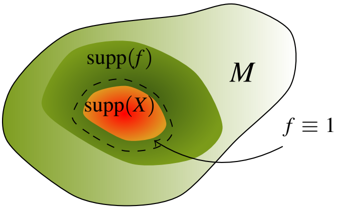

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- Functionals on $\mathfrak{E}_S(M)$: $\mathfrak{F}_S(M) \doteq \mathfrak{F}(M)/\mathfrak{F}_0(M) = H_0(\delta_S)$.

Koszul resolution

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- $H_0(\delta_S) = \mathfrak{F}(M)/\mathfrak{F}_0(M) = \mathfrak{F}_S(M)$,
- $H_k(\delta_S) = 0$, $k > 0$.

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- The so called *antibracket* is in our formalism just the Schouten bracket $\{.,.\}$ on the multivector fields.
- Derivation δ_S is not inner with respect to $\{.,.\}$, but locally it can be written as $\delta_S X = \{X, L_M(f)\}$ for $f \equiv 1$ on supp$X$, $X \in \mathfrak{V}(M)$.

Action and symmetries

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- We want to find gauge inv. functionals on-shell: $\mathfrak{F}^{\mathrm{inv}}_S$.

Ghosts and the Chevalley-Eilenberg complex

With the Lie algebra action of $\mathfrak{g}_c(M)$ on $\mathfrak{F}(M)$ one associates the Chevalley-Eilenberg complex. This is the graded algebra of smooth compactly supported multilocal (products of local) maps $\mathfrak{CE}(M) \doteq \mathcal{C}^\infty_{\mathrm{ml}}(\mathfrak{E}(M), \Lambda\mathfrak{g}'(M))$.

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and extended by continuity. In particular for $F\in\mathfrak{F}(M)$ we have: $(\gamma F)(X) = \partial_{\rho(X)}F$ and $\gamma F = 0$ if $F\in\mathfrak{F}^{\mathrm{inv}}(M)$.

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- Now we construct the Koszul-Tate resolution of the algebra $\mathfrak{CE}_S(M)$.

Koszul-Tate resolution

- Let $\mathfrak{BV}(M)$ be the subset of $S^\bullet \mathrm{Der}(\mathfrak{CE}(M))$ (graded symmetric powers) consisting of derivations that can be written as multilocal compactly supported maps on $\mathfrak{E}(M)$. The corresponding grading is denoted by $\#\mathrm{gh}$ (ghost number).

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- We define the BV differential: $sF = \{F, L_M(f) + \theta_M(f)\}$,

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- The full BV complex reads explicitely:

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- The gauge invariant observables are given by:

$$H^0(\mathfrak{B}\mathfrak{V}(M), s) = H^0(\mathfrak{C}\mathfrak{E}_S(M), s^{(0)}) = \mathfrak{F}^{\mathrm{inv}}_S(M)$$

BV complex on the fixed background

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On the fixed background the cohomology is trivial.

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Solution

We define the algebra of fields as: $Fld = \bigoplus_{k=0}^{\infty} \mathrm{Nat}(\mathfrak{E}_c^k, \mathfrak{B}\mathfrak{V})$. The action of symmetries on natural transformations $\Phi \in \mathrm{Nat}(\mathfrak{E}_c, \mathfrak{F})$:

$$(\rho_M(X)\Phi_M)(f) := \partial_{\rho_M(X)}(\Phi_M(f)) + \Phi_M(\rho_M(X)f), \qquad X \in \mathfrak{X}(M)\,.$$

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- In the framework of locally covariant field theory (Brunetti-Fredenhagen-Verch) fields are natural transformation between certain functors.

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- In classical gravity we understand physical quantities not as pointwise objects but rather as something defined on all the spacetimes in a coherent way.
- For example scalar curvature R is invariant in this sense, but $R(x)$ (curvature at a given point) not.

BV complex extended to natural transformations

- The set *Fld* becomes a graded algebra if we equip it with a graded product defined as:

$$(\Phi\Psi)_M(f_1,\dots,f_{p+q}) =$$
$$= \frac{1}{p!q!}\sum_{\pi\in P_{p+q}}\Phi_M(f_{\pi(1)},\dots,f_{\pi(p)})\Psi_M(f_{\pi(p+1)},\dots,f_{\pi(p+q)})\,.$$

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- The BV-differential on *Fld* is now given by:

$$(s\Phi)_M(f) := s_0(\Phi_M(f)) + (-1)^{|\Phi|}\Phi_M(\rho_M(.)f),$$

where s_0 is the BV differential on the fixed background.

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- The 0-cohomology of s is nontrivial, since it contains for example the Riemann tensor contracted with itself, smeared with a test function:

$$\Phi_{(M,g)}(f)(h) = \int_M R_{\mu\nu\alpha\beta}[\tilde{g}]R^{\mu\nu\alpha\beta}[\tilde{g}]f d\mathrm{vol}_{(M,\tilde{g})} \quad \tilde{g} = g + h\,.$$

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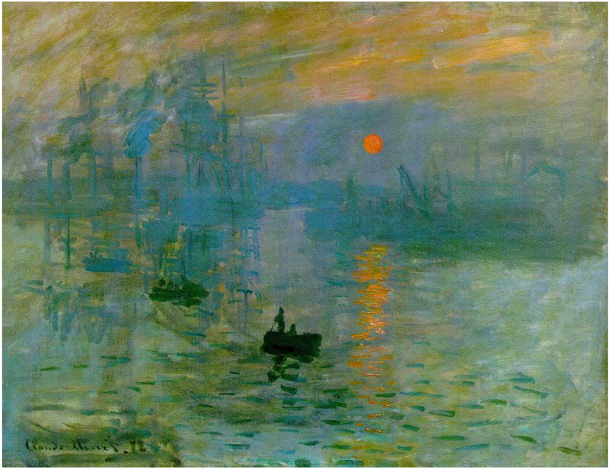
- The construction was formulated in a covariant way and generalized to the natural transformations.
- In general relativity the basic physical objects are fields (natural transformations), since they are defined not on a fixed background but rather on a class of spacetimes in a coherent way.
- The BV differential can be defined on the algebra of fields *Fld* and gives a homological interpretation to the notion of *gauge invariant physical quantities* in general relativity.



Thank you for your attention