1 A reminder of functional analysis

1.1 Proposition

A subset of a locally convex vector space (=:lcvs) is bounded if and only if every continuous, linear functional is bounded on it.

2 Talk 5

The main aim of this small talk is the definiton of convenient vector spaces with which I will shall start:

2.1 Theorem (Convenient vector space)

A lcvs E is called convenient or c^∞-complete if one of the following eqivalent conditions is satisfied:

(1) The RIEMANN integral exists for every LIPSCHITZ curve in E,

(2) for any $c \in \mathcal{C}^\infty(\mathbb{R}, E)$ there exists a $C \in \mathcal{C}^\infty(\mathbb{R}, E)$ with $C' = c$,

(3) E is c^∞-closed in any lcvs,

(4) if $c : \mathbb{R} \longrightarrow E$ is a curve such that $L \circ c : \mathbb{R} \longrightarrow \mathbb{R}$ is smooth $\forall L \in E^*$ (continuous, linear functionals on E), then c is smooth,

(5) every MACKEY-CAUCHY sequence converges; i.e. E is MACKEY complete,

(6) for any bounded, closed and absolutely convex set B is $E_B (:= \bigcup_{k \in \mathbb{K}} kB$ with norm $\|x\|_B = inf\{\lambda > 0 \mid x \in \lambda B\}$) a BANACH space, and

(7) any continuous, linear mapping from a normed space into E has a continuous extension to the completion of the normed space.

If I do not mention the bornology of an lcvs explicitly, I will always consider the von NEUMANN bornology. E always denotes a lcvs as well.
In this talk, I will focus on the conditions (1-4) that I would like to discuss in detail.
Therefore, I will directly tie in with Andreas' talk from last Friday that ended with the proof of the MACKEY convergence of the difference quotient.
An important consequence is the following:

2.2 Theorem (Smoothness of curves is a bornological concept)

For $0 \leq k \leq \infty$ a curve c in a lcvs E is $\mathcal{L}ip^k$
$:\Longleftrightarrow$ for each bounded, open intervall $I \subset \mathbb{R}$ exists an absolutely convex, bounded set $B \subseteq E$ such that $c\Big|_I$ is a $\mathcal{L}ip^k$-curve in the normed space E_B

proof:
„$\Longrightarrow$“: For k=0 this is an equivalent characteriziation of $\mathcal{L}ip$-curves: Take a bounded interval $I \subset \mathbb{R}$ and define B as the absolutely convex hull of the bounded set $c(I) \cup \{\frac{c(t)-c(s)}{t-s} \mid t \neq s, t, s \in I\}$ (finite union of bounded sets are bounded). Then $c\Big|_I : I \longrightarrow E_B$ is a well defined $\mathcal{L}ip$-curve in E_B.
For $k \geq 1$ chose a bounded intervall I and an absolutely convex set $B \subseteq E$ which contains all derivatives $c^{(i)}$ up to the order k as well as their difference quotients on $\{(t, s) \mid s \neq t, t, s \in I\}$.
c is differentiable, say at 0, with derivative $c'(0)$ which follows from $\frac{1}{t}(\frac{c'(t)-c(0)}{t} - c'(0)) \in B$ (see the proof of the MACKEY convergence of the different quotient). So $\frac{c'(t)-c(0)}{t} - c'(0)$ converges MACKEY to 0 in E and therefore $\frac{c'(t)-c(0)}{t} - c'(0)$ converges in E_B to 0 with respect to the norm topology.
The higher orders now follow by induction.

„$\Longleftarrow$“: This follows from the fact, that continuous, linear mappings between lcvs are smooth i.e. they map $\mathcal{L}ip^k$-curves to $\mathcal{L}ip^k$-curves.

This theorem shows that smoothness is really a bornological thing and does not depend on the topology but only on the dual since $c \in \mathcal{L}ip \iff L \circ c \in \mathcal{L}ip \ \forall L \in E^*$. So all topologies with the same dual have the same smooth curves. Furhtermore the class of $\mathcal{L}ip^k$-curves does not change if one passes from a given lct to its bornologification which is by definition the finest lct having the same bounded sets.
Now I want to give one further result of the MACKEY convergence of the difference quotient:

2.3 Lemma (Scalar testing of curves)

Let $c^k : \mathbb{R} \longrightarrow E$ for $0 \leq k < n+1$ curves such that $L \circ c^0 \in \mathcal{L}ip^n$ and $(L \circ c^0)^{(k)} = L \circ c^k \ \forall k, \forall L \in E^*$.
Then $c^0 \in \mathcal{L}ip^n$ and $(c^0)^{(k)} = c^k$.

To put this lemma to good use someone has to guess an appropriate candidate for the derivative.
One can ask now if someone always has to guess a candidate for the derivative in order to prove the convergence. In finite analysis on e.g. $\mathbb{R}$ this is not the case, since one can use the CAUCHY condition to show the convergence. This concept of CAUCHY nets and MACKEY-CAUCHY was introduced by Andreas last Friday.

2.4 Proposition (The difference quotient is MACKEY-CAUCHY)

Let $c : \mathbb{R} \longrightarrow E$ be a scalary (tested with the continuous, linear functionals) $\mathcal{L}ip^1$-curve in a lcvs E.
Then $t \longmapsto \frac{c(t)-c(0)}{t}$ is a MACKEY-CAUCHY net for $t \longrightarrow 0$

proof:
For a $\mathcal{L}ip^1$-curve this is an immediate consequence of the MACKEY convergence of the difference quotient. But here it is only assumed that $L \circ c$ is $\mathcal{L}ip^1$-curve $\forall L \in E^*$.
It suffices to show that $\frac{1}{t-s}\left(\frac{c(t)-c(0)}{t} - \frac{c(s)-c(0)}{s}\right)$ is bounded on bounded subsets of $\mathbb{R} \setminus \{0\}$. Due to 1.1 one can assume $E = \mathbb{R}$ and use the fundamental theorem of calculus:

$$\frac{1}{t-s}\Big(\frac{c(t)-c(0)}{t} - \frac{c(s)-c(0)}{s}\Big) = \int_0^1 \frac{c'(tr)-c'(sr)}{t-s} dr = \int_0^1 \frac{c'(tr)-c'(sr)}{tr-sr} r dr$$

which is locally bounded since $\frac{c'(tr)-c'(sr)}{tr-sr}$ is by assumption.

One consequence of this proposition is:

2.5 Lemma (Scalar testing of differentiable curves)

Let E be MACKEY complete and $c : \mathbb{R} \longrightarrow E$ be a curve for which $L \circ c \in \mathcal{L}ip^n \;\; \forall L \in E^*$.
Then $c \in \mathcal{L}ip^n$.

Here is another important general result dealing with linear maps and curves:

2.6 Lemma (Bounded linear maps)

A linear mapping $L : E \longrightarrow F$ between lcvs is bounded if and only if it is smooth which means that it maps smooth curves in E to smooth curves in F.

Now I will turn to the integration of curves.
One can show that for a continuous curve $c : [0, 1] \longrightarrow E$ the RIEMANN sums $R(c, Z) := \sum_{k=1}^{n} (t_k - t_{k-1})c(x_k)$ form a CAUCHY net with respect to the partial strict odering given by the size of the mesh $\mu(Z) := max\{|t_k - t_{k-1}| \mid 0 < k < n\}$, where $0 = t_0 < t_1 < \cdots < t_n = 1$ is a partition Z of $[0, 1]$ and $x_k \in [t_k, t_{k-1}]$. This will be of important concern when I discuss the integral auf LIPSCHITZ curves. Foremost some statements about the integral of curves:

2.7 Proposition (Integral of continuous curves)

Let $c : \mathbb{R} \longrightarrow E$ be a curve into a lcvs E and $\overline{E}^M$ its MACKEY completion.
Then there ist a unique differentiable curve $\int c : \mathbb{R} \longrightarrow \overline{E}^M$ such that $(\int c)(0) = 0$ and $(\int c)' = c$.

2.8 Definiton (Definite integral)

For continuous curves $c : \mathbb{R} \longrightarrow E$ *the definite integral* $\int_a^b c \in \overline{E}^M$ *is given by*

$$\int_a^b c = (\int c)(b) - (\int c)(a).$$

2.9 Corollary (Properties of the integral)

For a continuous curves $c : \mathbb{R} \longrightarrow E$ *holds:*

(1) $L(\int_a^b c) = \int_a^b (L \circ c) \ \ \forall L \in E^*$,

(2) $\int_a^b c + \int_b^d c = \int_a^d c$,

(3) $\int_a^b (c \circ \varphi)\varphi' = \int_{\varphi(a)}^{\varphi(b)} c$ *for* $\varphi \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$,

(4) $\int_a^b c$ *lies in the closed, convex hull in* $\overline{E}^M$ *of the set* $\{(b-a)c(t) \mid a < t < b\} \subseteq E$,

(5) $\int_a^b : \mathcal{C}(\mathbb{R}, E) \longrightarrow \overline{E}^M$ *is linear, and*

(6) for each $\mathcal{C}^1$*-curve* $c : \mathbb{R} \longrightarrow E$

$$\int_a^b c' = c(b) - c(a) \ \ \textit{(fundamental theorem of calculus).}$$

2.10 Proposition (Integral of LIPSCHITZ curves)

Let $c : [0,1] \longrightarrow E$ *be a LIPSCHITZ curve into a MACKEY complete lcvs* E*. Then the RIEMANN integral exists in* E *as the MACKEY limit of the RIEMANN sums.*

Proof: Let $0 < \epsilon \le 1$ and Z be a partition of $[0,1]$ with mesh $\mu(Z) \le \epsilon$ and refinement Z'. Let $[a,b]$ be an interval from the partition Z, $t \in [a,b]$ and $a = t_0 < t_1 < \cdots < t_n = b$ the refinement.

$$|b-a| \le \epsilon \Longrightarrow |t - t_k| \le \epsilon, \ \text{ for } \ 0 \le k \le n.$$

$$(b-a)c(t) - \sum_{k=1}^{n}(t_k - t_{k-1})c(t_k) = \sum_{k=1}^{n}(t_k - t_{k-1})\big(c(t) - c(t_k)\big) = \sum_{k=1}^{n}\mu_k b_k,$$

where $\mu_k = \epsilon \cdot (t_k - t_{k-1}) \geq 0$ with $\sum\limits_{k=1}^{n} \mu_k = (b-a)\epsilon$ and $b_k := \frac{c(t)-c(t_k)}{\epsilon}$ is contained in the absolutely convex, bounded set $B := abs.conv.Spann\Big(\{\frac{c(t)-c(s)}{t-s} \mid s,t \in [0,1]\}\Big)$. B is bounded since $c \in \mathcal{L}ip$.
It follows that

$$\begin{aligned}\frac{R(c,Z)-R(c,Z')}{\epsilon} &= \frac{1}{\epsilon}\sum_{l=1}^{m}\Big((b_l-a_l)c(t_l) - \sum_{k=1}^{n}(t_{kl}-t_{(k-1)l})c(t_{kl})\Big) \\ &= \sum_{l=1}^{m}\sum_{k=1}^{n}\underbrace{\frac{1}{\epsilon}(t_{kl}-t_{(k-1)l})}_{=:\mu_{kl}}\underbrace{\big(c(t_l)-c(t_{kl})\big)}_{=:b_{kl}\in B},\end{aligned}$$

where $\sum\limits_{l=1}^{m}\sum\limits_{k=1}^{n}\mu_{kl} = \sum\limits_{l=1}^{m}(b_l - a_l) = 1$ from which it follows that $R(c,Z) - R(c,Z') \in \epsilon B$.
Now take two partitions Z_1 and Z_2 of $[0,1]$ with mesh $\mu(Z_1) \leq \epsilon_1 \leq \epsilon$ and $\mu(Z_2) \leq \epsilon_2 \leq \epsilon$. Let Z be a common refinement (picture!) of Z_1 and Z_2, then

$$\begin{aligned}\frac{R(c,Z_1)-R(c,Z_2)}{2} &= \frac{R(c,Z_1)-R(c,Z)+R(c,Z)-R(c,Z_2)}{2} \\ &= \underbrace{\underbrace{\frac{1}{2}\big(R(c,Z_1)-R(c,Z)\big)}_{\in\epsilon B} - \underbrace{\frac{1}{2}\big(R(c,Z_2)-R(c,Z)\big)}_{\in\epsilon B}}_{\in\epsilon B}\end{aligned}$$

$\Longrightarrow R(c,Z_1) - R(c,Z_2) \in 2\epsilon B.$

So the RIEMANN sums for a LIPSCHITZ curve form a MACKEY-CAUCHY net with coefficients $\mu_{Z,Z'} := 2max\{\mu(Z),\mu(Z')\}$ and since E is MACKEY complete, they do converge.

2.11 Definition (c^∞-topology)

The c^∞-topology on a lcvs E is the final topology with respect to all smooth curves $\mathbb{R} \longrightarrow E$. In other words the c^∞-topology is the finest topology on E such that all smooth curves $\mathbb{R} \longrightarrow E$ become continuous.
The open sets of the c^∞-topology will be called c^∞-open.

The c^∞-topology will be treated in another talk in more detail but I would like to anticipate the following fact:
The finest lct coarser than the c^∞-topology ist the bornologification of the lcvs.

2.12 Convenient vector space

Finally I will return to the definition 2.1 of a convenient vector space and sketch a few easy implications:

(5)$\Longrightarrow$(4) is precisely lemma 2.5, (5)$\Longrightarrow$(1) is shown by proposition 2.10.

(1)$\Longrightarrow$(2): A smooth curve is LIPSCHITZ and thus locally RIEMANN integrable. The indefinite RIEMANN integral equals the intergal of proposition 2.7.

(3)$\Longrightarrow$(5): Let F be the MACKEY completion of E. Any MACKEY-CAUCHY sequence in E has a limit in F and since E is by assumption c^∞-closed in F the limit lies in E.