1 Reminder: Convergence of nets in topological spaces

Definition 1.1. *A directed system is an index set I together with an ordering $\prec$ which satisfies:*

1. *If $\alpha, \beta \in I$, then there exists $\gamma \in I$ sth. $\gamma \succ \alpha$ and $\gamma \succ \beta$*

2. *$\prec$ is a partial ordering (i.e. a reflexive transitive and antisymmetric relation on I)*

Definition 1.2. *A net in a topological space S is a mapping from a directed system I to S (notation: $(x_\alpha)_{\alpha \in I}$).*

Definition 1.3. *A net $(x_\alpha)_{\alpha \in I}$ in a topological space S is said to converge to $x \in S$ (notation: $x_\alpha \to x$) if for any neighborhood N oF x there is a $\beta \in I$ s.th. $x_\alpha \in N$ if $\alpha \succ \beta$.*

2 bornological convergence of nets

In a bornological vector space (bvs) E one has a natural notion of convergence (which depends only on the bornology $\mathcal{B}$). In many applications one uses convex bornological vector spaces (cbvs).

Definition 2.1. *Let $(x_\gamma)_{\gamma \in \Gamma}$ be a net in a cbvs E. We say that $(x_\gamma)_{\gamma \in \Gamma}$ converges bornologically to 0 ($(x_\gamma)_{\gamma \in \Gamma} \to 0$) if there exists a bounded and absolutely convex set $B \subset E$ and a net $(\lambda_\gamma)_{\gamma \in \Gamma}$ in $\mathbb{K}$ converging to 0 sth. $x_\gamma \in \lambda_\gamma B$.*

Correspondingly, $(x_\gamma)_{\gamma \in \Gamma}$ is said to converge bornologically to $x \in E$ if $(x_\gamma)_{\gamma \in \Gamma} - x \to 0$. Recall that absolutely convex is equivalent to disked. Furthermore, we define the vector space E_B wrt. the disk $B \subset E$ to be the linear span of B, which is equivalent to

$$E_B = \bigcup_{\lambda \in \mathbb{K}} \lambda B.$$

This space is then equipped with the seminorm $p_B(x) = \inf\{\alpha \in \mathbb{R}_+ | x \in \alpha B\}$, inducing a topology on E_B. If E is a lcvs and B is bounded additionally, the p_B is a norm.

Proposition 2.2 (characterisation of bornological convergence)**.** *Let $(x_\gamma)_{\gamma \in \Gamma}$ be a net in a cbvs E. Then $(x_\gamma)_{\gamma \in \Gamma} \to 0$ if and only if there exists a bounded absolutely convex set $B \subset E$ s.th. $(x_\gamma)_{\gamma \in \Gamma}$ converges to 0 in E_B (by which we mean topological convergence).*

Convention: If E is a topological vector space, then "$\to$" will denote topological convergence while "$\xrightarrow{M}$" (called Mackey-convergence) refers to bornological convergence wrt. the canonical von Neumann bornology.

Remark 2.3. *Let E be a lcvs and $B \subset E$ absolutely convex and bounded. Then the canonical embedding $E_B \to E$ is continous, so bornologically convergent nets (which converge topologically in E_B) converge also topologically in E. Generally the converse is false, as seen in the following*

Example 2.4. *Denote by* c_0 *the space of sequences converging to* 0 *and consider the space* $E = \prod_{c_0} \mathbb{R}$*, endowed with the product topology (which is the topology of pointwise convergence). Define* $x_n \in E$ *by its components* $(x_n)_\mu := \mu(n)$*. Clearly* (x_n) *converges to* 0 *wrt. this topology because it does so in every component. We show that* (x_n) *is not Mackey convergent: Suppose there is* $B \subset E$*, bounded and a sequence of reals* (λ_n) *converging to infinity sth.* $\{\lambda_n x_n : n \in \mathbb{N}\} \subseteq B \Leftrightarrow x_n \in 1/\lambda_n B$*. Project this on the component* κ*, given by* $\kappa_n := 1/\sqrt{\lambda_n} \in c_0$*. Thus* $\{\sqrt{\lambda_n} : n \in \mathbb{N}\} \subseteq pr_\kappa(B) \Rightarrow$ *Contradiction, since* $pr_\kappa(B)$ *must be bounded in* $\mathbb{R}$*. Thus* (x_n) *cannot be Mackey convergent since* B *and* (λ_n) *were arbitrary.*

Definition 2.5. *A net* $(x_\gamma)_{\gamma\in\Gamma}$ *in a cbvs is called Cauchy net if the net*

$$(x_\gamma - x_{\gamma'})_{(\gamma,\gamma')\in\Gamma\times\Gamma}$$

converges to 0*.*

Definition 2.6. *Let* E *be a separated topological vector space.* $(x_\gamma)_{\gamma\in\Gamma}$ *is called Mackey-Cauchy net if it is Cauchy wrt. the von Neumann bornology of* E*, i.e. there exists* $(\mu_{\gamma,\gamma'})_{(\gamma,\gamma')\in\Gamma\times\Gamma}$ *in* $\mathbb{R}$ *converging to* 0 *and* $B \subset E$*, bounded and absolutely convex s.th.* $(x_\gamma - x_{\gamma'}) \in \mu_{\gamma,\gamma'} B$*.*

Lemma 2.7.

1. *Let* E, F *be cbvs* $f : E \to F$ *be a bounded map. Let further* $x_\gamma \to x, y_\gamma \to y$ *in* E *and* $\lambda_\gamma \to \lambda$ *in* $\mathbb{K}$*. Then* $x_\gamma + y_\gamma \to x + y, \lambda_\gamma x_\gamma \to \lambda x$ *and* $f(x_\gamma) \to f(x)$*.*

2. *In a lcvs every Mackey convergent net is topologically convergent and every Mackey-Cauchy net is a Cauchy net.*

3. *In a lcs every weakly convergent Mackey-Cauchy net is Mackey convergent.*

Finally we can make a statement about the uniqueness of bornologically convergent nets in separated cbvs (recall that in a separated bornology $\{0\}$ is the only bounded vector subspace):

Proposition 2.8. *A cbvs is separated iff every convergent net has a unique limit.*

3 Completeness

Similarly to topological notions, one defines a bornological space to be complete if every bornological Cauchy sequence converges. In particular

Definition 3.1. *A lcvs* E *in which every Mackey-Cauchy sequence converges bornologically is called Mackey complete.*

Proposition 3.2. *For a lcvs* E *the following conditions are equivalent:*

1. *Every Mackey-Cauchy net converges topologically in* E

2. *Every Mackey-Cauchy sequence converges topologically in E*

3. *For every absolutely convex closed bounded set B the space E_B is complete*

4. *For every bounded set B there exists an absolutely convex bounded set $B' \supseteq B$ s.th. $E_{B'}$ is complete.*

Proof. 1.$\Rightarrow$ 2. and 3.$\Rightarrow$ 4. are clear.
2.$\Rightarrow$ 3.: Let (x_n) be Cauchy in E_B. Since E_B is normed, it suffices to show sequential completeness. By prop. 2.2, (x_n) is Mackey-Cauchy in E and converges to some $x \in E$ by assumption. Since $p_B(x_n - x_m) \to 0$, given $\epsilon > 0$ we find $N(\epsilon) \in \mathbb{N}$ s.th. $p_B(x_n - x_m) < \epsilon$ whenever $n, m > N(\epsilon)$ and thus $x_n - x_m \in \epsilon B$. Now $x_n - x \in \epsilon B$ for all $n > N(\epsilon)$ since B is closed. In particular $x \in E_B$ and thus $x_n \to x$ in E_B.
4.$\Rightarrow$ 1.: Let $(x_\gamma)_{\gamma\in\Gamma}$ be Mackey-Cauchy in E. There is some $\mu_{\gamma,\gamma'} \to 0$ in $\mathbb{R}$ s.th. $(x_\gamma - x_{\gamma'}) \in \mu_{\gamma,\gamma'} B$ for some B bounded. Let γ_0 be arbitrary and choose B to be absolutely convex, to contain x_{γ_0} and s.th. E_B is complete by (4.). For $\gamma \in \Gamma$ we have $x_\gamma = x_{\gamma_0} + x_\gamma - x_{\gamma_0} \in x_{\gamma_0} + \mu_{\gamma,\gamma_0} B \subset E_B$ and $p_B(x_\gamma - x_{\gamma'}) \leq \mu_{\gamma,\gamma'} \to 0$. Thus (x_γ) is Cauchy in E_B and converges in E_B and therefore in E. □

The following proposition establishes the equivalence of Mackey convergence and topological convergence in lcvs:

Proposition 3.3. *In a lcvs a Mackey-Cauchy net converges bornologically in E (i.e. E is Mackey complete) iff it converges topologically in E.*

Remark 3.4. *Since Mackey-Cauchy sequences of a lcvs are special Cauchy sequences, it follows from the last proposition and the equivalence 1.$\Leftrightarrow$2. before that a sequentially complete lcvs is Mackey complete, so Mackey completeness is a weaker condition. Example: space of distributions*

4 Lipschitz curves and Mackey convergence of the difference quotient

Definition 4.1. *Let E be a lcvs. A curve $c : \mathbb{R} \to E$ is called differentiable if the derivative $c'(t) := \lim_{s\to 0}(c(t+s) - c(t))/s$ at t exists for all t. c is called smooth or C^∞ if all iterated derivatives exist. It is called C^n for $n < \infty$ if its iterated derivatives up to order n exist and are continous.*

Definition 4.2. *A curve $c : \mathbb{R} \to E$ is called locally Lipschitzian if every point $r \in \mathbb{R}$ has a neighborhood U sth. the Lipschitz codition is satisfied on U, i.e. the set $\{\frac{1}{t-s}(c(t) - c(s)) : t \neq s; t, s \in U\}$ is bounded.*

This implies that for c the Lipschitz condition is satisfied on each bounded interval since for increasing t_i

$$\frac{c(t_n) - c(t_0)}{t_n - t_0} = \sum \frac{t_{i+1} - t_i}{t_n - t_0} \frac{c(t_{i+1}) - c(t_i)}{t_{i+1} - t_i}$$

lies in the absolutely convex hull of a finite union of bounded sets. $c : \mathbb{R} \to E$ is called $\mathcal{L}ip^k$ if all derivatives up to order k exist and are locally Lipschitzian.

4.1 Mean value theorem

Motivation: For curves c with values in a finite dimensional space there is a generalised version of the mean value theorem in one dimension, namely for an additional function $h : \mathbb{R} \to \mathbb{R}$ with nonvanishing derivative we have that $\frac{c(a)-c(b)}{h(a)-h(b)}$ lies in the closed convex hull of $\{c'(r)/h'(r) : r\}$

Proposition 4.3. *Let $c : I := [a, b] \to E$ be a continous curve which is differentiable except at points in a countable subset $D \subseteq I$. Let h be a continous monotone function $h : I \to \mathbb{R}$ which is differntiable on $I \setminus D$. Let A be a convex closed subset of E sth. $c'(t) \in h'(t)A$ for all $t \notin D$. Then $c(b) - c(a) \in (h(b) - h(a))A$.*

4.2 The difference quotient converges Mackey

Proposition 4.4. *Let $c : \mathbb{R} \to E$ be a $\mathcal{L}ip^1$-curve. Then the curve $\frac{1}{t}\left(\frac{1}{t}(c(t) - c(0)) - c'(0)\right)$ is bounded on subsets of $\mathbb{R} \setminus \{0\}$.*

Proof. Apply 4.3 with $h = \mathrm{Id}$ to c and obtain:

$$\begin{aligned}\frac{c(t)-c(0)}{t} - c'(0) &\in \langle c'(r) : 0 < |r| < |t|\rangle_{\text{closed,convex}} - c'(0)\\ &= \langle c'(r) - c'(0) : 0 < |r| < |t|\rangle_{\text{closed,convex}}\\ &= \left\langle r\frac{c'(r)-c'(0)}{r} : 0 < |r| < |t|\right\rangle_{\text{closed,convex}}\end{aligned}$$

Let $a > 0$. Since $\{\frac{c'(r)-c'(0)}{r} : 0 < |r| < |a|\}$ is bounded and hence contained in a closed absolutely convex and bounded set B it follows that

$$\frac{1}{t}\left(\frac{c(t)-c(0)}{t} - c'(0)\right) \in \left\langle \frac{r}{t}\frac{c'(r)-c'(0)}{r} : 0 < |r| < |t|\right\rangle_{\text{closed,convex}} \subseteq B$$

□