Cartesian closedness in convenient setting II

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1 Remainder

Definition 1. convenient vector space

Definition 2. c^∞-topology

Remark 1 (on c^∞-open sets)**.** *Let E be a bornological locally convex vector space, $U \subseteq E$ a convex subset. Then U is open for the locally convex topology of E if and only if U is open for the c^∞-topology. Furthermore, an absolutely convex subset U of E is a 0-neighborhood for the locally convex topology if and only if it is so for the c^∞-topology.*

Proof. ($\Rightarrow$) The c^∞-topology is finer than the locally convex topology.
($\Leftarrow$) Let first U be an absolutely convex 0-neighborhood for the c^∞-topology. Hence, U absorbs Mackey-0-sequences. Since a locally convex vector space is bornological if and only if every absolutely convex bornivorous subset is a 0-neighborhood, we have to show that U is bornivorous. Then it would follow that U is a 0-neighborhood for the locally convex topology.

That U is bornivorous follows from the fact that U absorbs all sequences converging Mackey to 0. This is a consequence of the simple lemma:

Lemma 3. *For a seminorm p the following statements are equivalent:*

1. *p is bounded;*
2. *p is bounded on compact sets;*
3. *p is bounded on M-converging sequences;*

Let now U be convex and c^∞-open, let $x \in U$ be arbitrary. We consider the c^∞- open absolutely convex set $W \doteq (U - x) \cap (x - U)$ which is a 0-neighborhood of the locally convex topology by the argument above. Then $x \in W + x \subseteq U$. So U is open in the locally convex topology. □

Theorem 4. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary mapping. Then all iterated partial derivatives exist and are locally bounded if and only if the associated mapping $f^\vee : \mathbb{R} \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ exists as a smooth curve, where $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ is considered as the Fréchet space with the topology of uniform convergence of each derivative on compact sets. Furthermore, we have $(\partial_1 f)^\vee = d(f^\vee)$ and $(\partial_2 f)^\vee = d \circ f^\vee$.*

Theorem 5. *[Boman, 1967] For a mapping $f : \mathbb{R}^2 \to \mathbb{R}$ the following assertions are equivalent:*

1. *All iterated partial derivatives exist and are continuous.*
2. *All iterated partial derivatives exist and are locally bounded.*
3. *For $v \in \mathbb{R}^2$ the iterated directional derivatives:*
$$d_v^n f(x) := \left(\frac{\partial}{\partial t}\right)^n \Big|_{t=0} (f(x+tv)) \tag{1}$$
exist and are locally bounded with respect to x.
4. *For all smooth curves $c : \mathbb{R} \to \mathbb{R}^2$ the composite $f \circ c$ is smooth.*

Definition 6. We define $\mathcal{C}^\infty(\mathbb{R}, E)$ to be the locally convex vector space of all smooth curves into E, with the pointwise vector operations, and with the topology of **uniform convergence on compact sets of each derivative separately**.

Theorem 7. *For a mapping $f : \mathbb{R}^2 \to E$ into a locally convex space (which need not be c^∞-complete) the following assertions are equivalent:*

1. *f is smooth along smooth curves.*
2. *All iterated directional derivatives $d_v^p f$ exist and are locally bounded.*
3. *All iterated partial derivatives $\partial_\alpha f$ exist and are locally bounded.*
4. *$f^\vee : \mathbb{R} \to \mathcal{C}^\infty(\mathbb{R}, E)$ exists as a smooth curve.*

Definition 8. A mapping $f : E \supseteq U \to F$ defined on a c^∞-open subset U is called smooth (or $\mathcal{C}^\infty$) if it maps smooth curves in U to smooth curves in F. By $\mathcal{C}^\infty(U, F)$ we shall denote the locally convex space of all smooth mappings $U \to F$ with pointwise linear structure and the initial topology with respect to all mappings $c^* : \mathcal{C}^\infty(U, F) \to \mathcal{C}^\infty(\mathbb{R}, F)$ for $c \in \mathcal{C}^\infty(\mathbb{R}, U)$.

Remark 2. *For $U = E = \mathbb{R}$ this coincides with our old definition.*

Theorem 9. *Let $U_i \subseteq E_i$ be c^∞-open subsets in locally convex spaces, which need not be c^∞-complete. Then a mapping $f : U_1 \times U_2 \to F$ is smooth if and only if the canonically associated mapping $f^\vee : U_1 \to \mathcal{C}^\infty(U_2, F)$ exists and is smooth.*

2 Consequences of cartesian closedness

Corollary 10. *Let $E, F, G, \dots$ be locally convex spaces, and let U, V be c^∞-open subsets of such. Then the following canonical mappings are smooth:*

1. *$ev : \mathcal{C}^\infty(U, F) \times U \to F$, $(f, x) \mapsto f(x)$*
2. *$ins : E \to \mathcal{C}^\infty(F, E \times F)$, $x \mapsto (y \mapsto (x, y))$*
3. *$(\bullet)^\wedge : \mathcal{C}^\infty(U, \mathcal{C}^\infty(V, G)) \to \mathcal{C}^\infty(U \times V, G)$*
4. *$(\bullet)^\vee : \mathcal{C}^\infty(U \times V, G) \to \mathcal{C}^\infty(U, \mathcal{C}^\infty(V, G))$*
5. *$comp : \mathcal{C}^\infty(F, G) \times \mathcal{C}^\infty(U, F) \to \mathcal{C}^\infty(U, G)$, $(f, g) \mapsto f \circ g$*

6. $\mathcal{C}^\infty(\bullet, \bullet) : \mathcal{C}^\infty(E_2, E_1) \times \mathcal{C}^\infty(F_1, F_2) \to \mathcal{C}^\infty(\mathcal{C}^\infty(E_1, F_1), \mathcal{C}^\infty(E_2, F_2))$,
   $(f, g) \mapsto (h \mapsto g \circ h \circ f)$

7. $\prod : \prod \mathcal{C}^\infty(E_i, F_i) \to \mathcal{C}^\infty(\prod E_i, \prod F_i)$, *for any index set.*

Proof. (1) The mapping associated to ev via cartesian closedness is the identity on $\mathcal{C}^\infty(U, F)$, which is $\mathcal{C}^\infty$, thus ev is also $\mathcal{C}^\infty$.
(2) The mapping associated to ins via cartesian closedness is the identity on $E \times F$, hence ins is $\mathcal{C}^\infty$.
(3) The mapping associated to $(.)^\wedge$ via cartesian closedness is the smooth composition of evaluations $ev \circ (ev \times \mathrm{Id}) : (f; x, y) \mapsto f(x)(y)$.
(4) We apply cartesian closedness twice to get the associated mapping $(f; x; y) \mapsto f(x, y)$, which is just a smooth evaluation mapping.
(5) The mapping associated to $comp$ via cartesian closedness is $(f, g; x) \mapsto f(g(x))$, which is the smooth mapping $ev \circ (\mathrm{Id} \times ev)$.
(6) The mapping associated to the one in question by applying cartesian closedness twice is $(f, g; h, x) \mapsto g(h(f(x)))$, which is the $\mathcal{C}^\infty$-mapping $ev \circ (\mathrm{Id} \times ev) \circ (\mathrm{Id} \times \mathrm{Id} \times ev)$.
(7) Up to a flip of factors the mapping associated via cartesian closedness is the product of the evaluation mappings $\mathcal{C}^\infty(E_i, F_i) \times E_i \to F_i$. □

Example 1 (after [1]). *Consider the evaluation $ev : E \times E^* \to \mathbb{R}$, where E is a locally convex space and E^* is it's dual of continuous linear functionals equipped with any locally convex topology. Let us assume that the evaluation is jointly continuous. Then there are neighborhoods $U \subseteq E$ and $V \subseteq E^*$ of zero such that $ev(U \times V) \subseteq [-1, 1]$. But then U is contained in the polar of V , so it is bounded in E, and so E admits a bounded neighborhood and is thus normable.*

Definition 11. Given a dual pair (X, Y) the polar set or polar of a subset A of X is a set A° in Y defined as:

$$A^\circ := \{y \in Y : \sup\{|\langle x, y\rangle| : x \in A\} \leq 1\} \tag{2}$$

Corollary 12 (Boman, 1967). *The smooth mappings on open subsets of R^n in the sense of definition 8 are exactly the usual smooth mappings.*

Proof. ($\Rightarrow$) If $f : \mathbb{R}^n \to F$ is smooth then by cartesian closedness 9, for each coordinate the respective associated mapping $f^{\vee_i} : \mathbb{R}^{n-1} \to \mathcal{C}^\infty(\mathbb{R}, F)$ is smooth, so again by 8 we have $\partial_i f = (d_* f^{\vee_i})^\wedge$, so all first partial derivatives exist and are smooth. Inductively, all iterated partial derivatives exist and are smooth, thus continuous, so f is smooth in the usual sense.
($\Leftarrow$) Obviously, f is smooth along smooth curves by the usual chain rule. □

Definition 13. By $L(E, F)$ we denote the space of all bounded (equivalently smooth) linear mappings from E to F . It is a closed linear subspace of $\mathcal{C}^\infty(E, F)$ since f is linear if and only if for all $x, y \in E$ and $\lambda \in \mathbb{R}$ we have $(ev_x + \lambda ev_y - ev_{x+\lambda y})f = 0$. We equip it with this topology and linear structure.

Theorem 14 (Chain rule). *Let E and F be locally convex spaces, and let $U \subseteq E$ be c^∞-open. Then the differentiation operator:*

$$d \quad : \quad \mathcal{C}^\infty(U, F) \to \mathcal{C}^\infty(U, L(E, F)), \tag{3}$$

$$df(x)v \quad := \quad \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}, \tag{4}$$

exists, is linear and bounded (smooth). Also the chain rule holds:

$$d(f \circ g)(x)v = df(g(x))dg(x)v. \tag{5}$$

Proof. We want to show that $d^{\wedge\wedge} : \mathcal{C}^\infty(U, F) \times U \times E \to F$ is smooth, so let $(f, x, v) : \mathbb{R} \to \mathcal{C}^\infty(U, F) \times U \times E$ be a smooth curve. Then

$$d^{\wedge\wedge}(f(t), x(t), v(t)) = \lim_{s\to 0} \frac{f(t)(x(t) + sv(t)) - f(t)(x(t))}{s} = \partial_2 h(t, 0), \tag{6}$$

which is smooth in t, where the smooth mapping $h : \mathbb{R}^2 \to F$ is given by $(t, s) \mapsto f^\wedge(t, x(t) + sv(t))$. By cartesian closedness 9 the mapping $d^\wedge : \mathcal{C}^\infty(U, F) \times U \to \mathcal{C}^\infty(E, F)$ is smooth. Now we show that this mapping has values in the subspace $L(E, F)$. It is easy to see that $d^\wedge(f, x) \in \mathcal{C}^\infty(E, F)$ is homogeneous. It is additive, because we may consider the smooth mapping $\tilde{f} : (t, s) \mapsto f(x + tv + sw)$ and compute as follows, using 12.

$$\begin{aligned} df(x)(v + w) &= d\tilde{f}(0, 0) = \frac{\partial}{\partial t}\tilde{f}(0, 0) + \frac{\partial}{\partial s}\tilde{f}(0, 0) = & (7)\\ &= \frac{\partial}{\partial t}\Big|_0 f(x + tv + 0w) + \frac{\partial}{\partial t}\Big|_0 f(x + 0v + tw) = df(x)v + df(x)w. & (8)\end{aligned}$$

So we see that $d^\wedge : \mathcal{C}^\infty(U, F) \times U \to L(E, F)$ is smooth, and the mapping $d : \mathcal{C}^\infty(U, F) \to \mathcal{C}^\infty(U, L(E, F))$ is smooth by 9 and obviously linear. We first prove the chain rule for a smooth curve c instead of g. We have to show that the curve

$$t \mapsto \begin{cases} \frac{f(c(t)) - f(c(0))}{t} & \text{for} \quad t \neq 0 \\ df(c(0)).c'(0) & \text{for} \quad t = 0 \end{cases} \tag{9}$$

is continuous at 0. It can be rewritten as $t \mapsto \int_0^1 df(c(0) + s(c(t) - c(0))).c_1(t)ds$, where c_1 is the smooth curve given by

$$t \mapsto \begin{cases} \frac{c(t) - c(0)}{t} & \text{for} \quad t \neq 0 \\ c'(0) & \text{for} \quad t = 0 \end{cases} \tag{10}$$

Since $h : \mathbb{R}^2 \to U \times E$ given by

$$(t, s) \mapsto (c(0) + s(c(t)c(0)), c_1(t)) \tag{11}$$

is smooth, the map $\tilde{h} = (d^{\wedge\wedge} \circ h)^\vee$ is also smooth. It is given explicitely by: $\tilde{h} : t \mapsto (s \mapsto df(c(0) + s(c(t) - c(0))).c_1(t))$. It is smooth map $\mathbb{R} \to \mathcal{C}^\infty(\mathbb{R}, F)$, and hence a map $\tilde{c} : t \mapsto \int \tilde{h}(t)$ is also smooth, and therefore continuous (since this is a smooth curve). This ends the proof of the chain rule for c, because $\tilde{c}$ is the curve from (9), namely: $\tilde{c} : t \mapsto \int_0^1 df(c(0) + s(c(t) - c(0))).c_1(t)ds$. For general g we have:

$$d(f \circ g)(x)(v) = \frac{\partial}{\partial t}\Big|_0 (f \circ g)(x + tv) \tag{12}$$

We can take $c(t) = g(x + tv)$ and use the previous result to conclude that:

$$\begin{aligned} d(f \circ g)(x)(v) &= d(f \circ c)(0).0 = df(c(0))c'(0) = & (13)\\ &= (df)(g(x + 0v)) \left(\frac{\partial}{\partial t}\Big|_0 (g(x + tv))\right) = (df)(g(x))(dg(x)(v)) & (14)\end{aligned}$$

□

3 Bounded multilinear mappings

Why take $L(E,F)$ as bounded linear mappings? The reason for this is the **Uniform Boundedness Principle**. It is very important for convenient calculus. Essentially it says that on $L(E,F)$ the canonical and pointwise bornologies concide if E, F are convenient vector spaces. As a remeinder:

Definition 15. The **canonical bornology** has as bounded sets $\mathcal{B} \subseteq L(E,F)$ such that $\mathcal{B}(A)$ is bounded in F for all A, bounded in E.
The **pointwise bornology** has as bounded sets $\mathcal{B} \subseteq L(E,F)$ which satisfy $\mathcal{B}(a)$ bounded in F for all $a \in E$.

In other words, there is essentially one bornology on $L(E,F)$ but different, non-equivalent topologies. The category of all smooth mappings between bornological vector spaces is a subcategory of the category of all smooth mappings between locally convex spaces which is equivalent to it, since a locally convex space and its bornologification have the same bounded sets and smoothness depends only on the bornology. So it is also cartesian closed, but the topology on $\mathcal{C}^\infty(E,F)$ from 8 has to be bornologized. The next example shows that indeed, this topology is in general not bornological.

Example 2 (after [2], 5.4.19)**.** *The locally convex topology of* $\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{(\mathbb{N})})$ *is not bornological.*

Proof. In order to see this we consider the linear functional $\ell : \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{(\mathbb{N})}) \to \mathbb{R}$ defined by $\ell(f) \doteq \sum_{k\in\mathbb{N}} (pr_k \circ f)^{(k)}(0)$. For any bounded subset $B \subseteq \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{(\mathbb{N})})$ there exists an $N \in \mathbb{N}$ such that $B \subseteq \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^N)$. Hence on such a set B the functional ℓ is a fnite sum of derivatives at 0 composed with projections pr_k and thus is a bornological (bounded) map. But ℓ cannot be continuous w.r.t. locally convex topology of $\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{(\mathbb{N})})$, because otherwise there would exist an $N \in \mathbb{N}$ and a 0-neighbourhood $U \subseteq \mathbb{R}^{(\mathbb{N})}$ s.t. $f^{(k)}(t) \in U$ fot $k < N$ and $|t| \leq N$ would imply $|\ell(f)| \leq 1$. This is impossible, since among all functions f satisfying $f^{(k)}(t) \in U$ for $k < N$ and $|t| \leq N$ there are such with only the projection $f_N \doteq pr_N \circ f$ unequal 0 and N-th derivative of f_N at 0 larger than 1. □

Proposition 16. *Exponential law for L. There are natural bornological isomorphisms:*

$$L(E_1,\dots,E_{n+k};F) \cong L(E_1,\dots,E_n;L(E_{n+1},...,E_{n+k};F)) \tag{15}$$

Proof. (for bilinear mappings, the general case is completely analogous). Bilinearity translates into linearity into the space of linear functions. It remains to prove boundedness. So let $\mathcal{B} \subseteq L(E_1,E_2;F)$ be given. Then $\mathcal{B}$ is bounded if and only if $\mathcal{B}(B_1 \times B_2) \subseteq F$ is bounded for all bounded $B_i \subseteq E_i$. This however is equivalent to $\mathcal{B}^\vee(B_1)$ is contained and bounded in $L(E_2,F)$ for all bounded $B_1 \subseteq E$, i.e. $\mathcal{B}^\vee$ is contained and bounded in $L(E_1,L(E_2,F))$. □

Lemma 17. *A subset is bounded in* $L(E,F) \subseteq \mathcal{C}^\infty(E,F)$ *if and only if it is uniformly bounded on bounded subsets of* E*, i.e.* $L(E,F) \to \mathcal{C}^\infty(E,F)$ *is initial.*

Proof. Let $B \subseteq L(E,F)$ be bounded in $\mathcal{C}^\infty(E,F)$, and assume that it is not uniformly bounded on some bounded set $B \subseteq E$. So there are $f_n \in \mathcal{B}$, $b_n \in B$, and $\ell \in F^*$ with $|\ell(f_n(b_n))| \geq n^n$.

Definition 18. We say that a sequence x_n in a locally convex space E converges fast to x in E, if for each $k \in \mathbb{N}$ the sequence $n^k(x_n - x)$ is bounded.

Lemma 19. *(Special curve lemma) Let x_n be a sequence which converges fast to x in E. Then the infinite polygon through the x_n can be parameterized as a smooth curve $c : \mathbb{R} \to E$ such that $c(\frac{1}{n}) = x_n$ and $c(0) = x$.*

The sequence $n^{1-n}b_n$ converges fast to 0, and hence lies on some compact part of a smooth curve c by the special curve lemma. So B cannot be bounded, since otherwise $\mathcal{C}^\infty(\ell, c) = \ell_* \circ c^* : \mathcal{C}^\infty(E, F) \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \to \ell^\infty(\mathbb{R}, \mathbb{R})$ would have bounded image, i.e. $\{\ell \circ f_n \circ c : n \in \mathbb{N}\}$ would be uniformly bounded on any compact interval.
Conversely, let $B \subseteq L(E, F)$ be uniformly bounded on bounded sets and hence in particular on compact parts of smooth curves. We have to show that $d^n \circ c^* : L(E, F) \to \mathcal{C}^\infty(\mathbb{R}, F) \to \ell^\infty(\mathbb{R}, F)$ has bounded image. But for linear smooth maps we have by the chain rule, recursively applied, that $d^n(f \circ c)(t) = f(c^{(n)}(t))$, and since $c^{(n)}$ is a smooth curve, the conclusion follows. □

Lemma 20. *(Bounded multilinear mappings are smooth). Let $f : E_1 \times \ldots \times E_n \to F$ be a multilinear mapping. Then f is bounded if and only if it is smooth. For the derivative we have the product rule:*

$$df(x_1, \ldots, x_n)(v_1, \ldots, v_n) = \sum_{i=1}^{n} f(x_1, \ldots, x_{i-1}, v_i, x_{i+1}, \ldots, x_n) \tag{16}$$

In particular, we get for $f : E \supseteq U \to \mathbb{R}$, $g : E \supseteq U \to F$ and $x \in U$, $v \in E$ the Leibniz formula

$$(f \cdot g)'(x)(v) = f'(x)(v) \cdot g(x) + f(x) \cdot g'(x)(v) \tag{17}$$

Proof. We use induction on n. The case $n = 1$ follows, since a linear mapping $E \to F$ between locally convex vector spaces is bounded (or bornological) if and only if it maps smooth curves in E to smooth curves in F. The induction goes as follows:
f is bounded $\Leftrightarrow$ $f(B_1 \times \ldots \times B_n) = f^\wedge(B_1 \times \ldots \times B_{n-1})(B_n)$ is bounded for all bounded sets B_i in E_i $\Leftrightarrow$ $f^\vee(B_1 \times \ldots \times B_{n-1}) \subseteq L(E_n, F) \subseteq \mathcal{C}^\infty(E_n, F)$ is bounded, by 17; $\Leftrightarrow$ $f^\vee : E_1 \times \ldots \times E_{n-1} \to \mathcal{C}^\infty(E_n, F)$ is bounded; $\Leftrightarrow$ $f^\vee : E_1 \times \ldots \times E_{n-1} \to \mathcal{C}^\infty(E_n, F)$ is smooth by the inductive assumption; $\Leftrightarrow$ $f : E_1 \times \ldots \times E_n \to F$ is smooth by cartesian closedness
. □

4 Spaces of smooth mappings

Proposition 21. *Let M be a smooth finite dimensional paracompact manifold. Then the space $\mathcal{C}^\infty(M, \mathbb{R})$ of all smooth functions on M is a convenient vector space in any of the following (bornologically) isomorphic descriptions:*

1. *The initial structure with respect to the cone*

$$\mathcal{C}^\infty(M, \mathbb{R}) \xrightarrow{c^*} \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \tag{18}$$

   for all $c \in \mathcal{C}^\infty(\mathbb{R}, M)$.

2. *The initial structure with respect to the cone*

$$\mathcal{C}^\infty(M, \mathbb{R}) \xrightarrow{(u_\alpha^{-1})^*} \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})\,, \tag{19}$$

   where (U_α, u_α) is a smooth atlas with $u_\alpha(U_\alpha) = \mathbb{R}^n$.

3. *The initial structure with respect to the cone*

$$\mathcal{C}^\infty(M,\mathbb{R}) \xrightarrow{j^k} C(J^k(M,\mathbb{R})) \tag{20}$$

for all $k \in \mathbb{N}$, where $J^k(M,\mathbb{R})$ is the bundle of k-jets of smooth functions on M , where j^k is the jet prolongation, and where all the spaces of continuous sections are equipped with the compact open topology.

It is easy to see that the cones in (2) and (3) induce even the same locally convex topology which is sometimes called the compact $\mathcal{C}^\infty$ topology, if $\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})$ is equipped with its usual Fréchet topology.

For a smooth separable finite dimensional Hausdorff manifold M we denote by $\mathcal{C}^\infty_c(M,\mathbb{R})$ the vector space of all smooth functions with compact supports in M.

Proposition 22. *The following convenient structures on the space $\mathcal{C}^\infty_c(M,\mathbb{R})$ are all isomorphic:*

1. *Let $\mathcal{C}^\infty_K(M,\mathbb{R})$ be the space of all smooth functions on M with supports contained in the fixed compact subset $K \subseteq M$, a closed linear subspace of $\mathcal{C}^\infty(M,\mathbb{R})$. Let us consider the final convenient vector space structure on the space $\mathcal{C}^\infty_c(M,\mathbb{R})$ induced by the cone*

$$\mathcal{C}^\infty_K(M,\mathbb{R}) \to \mathcal{C}^\infty_c(M,\mathbb{R}) \tag{21}$$

where K runs through a basis for the compact subsets of M . Then the space $\mathcal{C}^\infty_c(M,\mathbb{R})$ is even the strict inductive limit of a sequence of spaces $\mathcal{C}^\infty_K(M,\mathbb{R})$.

2. *We equip $\mathcal{C}^\infty_c(M,\mathbb{R})$ with the initial structure with respect to the cone:*

$$\mathcal{C}^\infty_c(M,\mathbb{R}) \xrightarrow{e^*} \mathcal{C}^\infty_c(\mathbb{R},\mathbb{R})\,, \tag{22}$$

where $e \in \mathcal{C}^\infty_{prop}(\mathbb{R},M)$ runs through all proper smooth mappings $\mathbb{R} \to M$, and where $\mathcal{C}^\infty_c(\mathbb{R},\mathbb{R})$ carries the usual inductive limit topology on the space of test functions, with steps $\mathcal{C}^\infty_I(\mathbb{R},\mathbb{R})$ for compact intervals I.

3. *The initial structure with respect to he cone*

$$\mathcal{C}^\infty_c(M,\mathbb{R}) \xrightarrow{j^k} C_c(J^k(M,\mathbb{R})) \tag{23}$$

for all $k \in \mathbb{N}$, where $J^k(M,\mathbb{R})$ is the bundle of k-jets of smooth functions on M , where j^k is the jet prolongation, and where the spaces of continuous sections with compact support are equipped with the inductive limit topology with steps $C_K(J^k(M,\mathbb{R}))$.

5 Remarks on c^∞-topology

The c^∞-topology of a product. Consider the product $E \times F$ of two locally convex vector spaces. Since the projections onto the factors are linear and continuous, and hence smooth, we always have that the identity mapping $c^\infty(E \times F) \to c^\infty(E) \times c^\infty(F)$ is continuous. It is not always a homeomorphism: Just take a bounded separately continuous bilinear functional, which is not continuous (like the evaluation map) on a product of spaces where the c^∞-topology is the bornological topology. However, if one of the factors is finite dimensional the product is well behaved:

Proposition 23. *For any locally convex space E the c^∞-topology of $E \times \mathbb{R}^n$ is the product topology of the c^∞-topologies of the two factors, so that we have $c^\infty(E \times \mathbb{R}^n) = c^\infty(E) \times \mathbb{R}^n$.*

Proposition 24. *Let E and F be bornological locally convex vector spaces. If there exists a bilinear smooth mapping $m : E \times F \to \mathbb{R}$ that is not continuous with respect to the locally convex topologies, then $c^\infty(E \times F)$ is not a topological vector space.*

Corollary 25. *Let E be a non-normable bornological locally convex space. Then $c^\infty(E \times E')$ is not a topological vector space.*

References

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