Infinite dimensional differential geometry seminar

February 19, 2010

1 Introduction

1.1 Why is ∞-dimensional interesting?

1. Mathamatics:

   - Calculus beyond Banach spaces
   - Infinite dimensional manifolds
   - Infinite dimensional Lie groups:
     - infinite-dimensional Lie theory
     - homotopy groups
     - extensions

2. Physics:

   - Classical field theory: variational calculus, functional derivatives, space of local functionals
   - Quantum field theory: various spaces of functionals, functional derivatives, notion of convergence in the space of functional (what is a "nice" topology?).
   - Infinite dimensional Lie groups: gauge group, diffeomorphism group of a manifold.

1.2 Functional analysis remainder

Definition 1. A **topological space** is a set X in which a collection τ of subsets (called open sets) has been specified, with the following properties:

- $X \in \tau$
- $\emptyset \in \tau$
- the intersection of any two open sets is open: $U \cap V \in \tau$ for $U, V \in \tau$
- the union of every collection of open sets is open: $\bigcup_{\alpha \in A} U_\alpha \in \tau$ for $U_\alpha \in \tau \; \forall \alpha \in A$.

Definition 2. A function $f : X \to Y$, where X and Y are topological spaces, is **continuous** if and only if for every open set $V \subseteq Y$, the inverse image:

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\} \tag{1}$$

is open.

Definition 3. **Topological vector space**: a pair (X, τ), where τ is a topology on a vector space X such that:

- every point of X is a closed set
- the vector space operations are continuous with respect to τ.

Definition 4. Let E be a vector space over a field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and $A, B \subseteq E$:

1. A is called **circled** if $\forall \lambda \in \mathbb{K}, |\lambda| \leq 1 : \lambda A \subseteq A$.
2. A is called **balanced** if $\forall \lambda \in \mathbb{K}, |\lambda| = 1 : \lambda A \subseteq A$.
3. A is said to **absorb** B if there exists a $\lambda > 0$ with $[0, \lambda] \cdot B \subseteq A$.
4. A is called **absorbent** if $\forall x \in E : A$ absorbs $\{x\}$.
5. A is called **convex** if $\mathbb{R} \ni \lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$, $x_1, x_2 \in A$ implies: $\lambda_1 x_1 + \lambda_2 x_2 \in A$.
6. A is called **absolutely convex** if $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$. $\sum_{i=1}^{n} |\lambda_i| \leq 1$, $x_1, \ldots, x_n \in A$ implies $\sum_{i=1}^{n} \lambda_i x_i \in A$.

Definition 5. A **local base** of a topological vector space X is thus a collection $\mathscr{B}$, of neighborhoods of 0 such that every neighborhood of 0 contains a member of $\mathscr{B}$. The open sets of X are then precisely those that are unions of translates of members of $\mathscr{B}$.

Definition 6. Types of topological vector spaces. In the following definitions, X always denotes a topological vector space, with topology τ.

1. X is locally convex if there is a local base $\mathscr{B}$ whose members are convex.
2. X is locally bounded if 0 has a bounded neighborhood.
3. X is locally compact if 0 has a neighborhood whose closure is compact.
4. X is metrizable if τ is compatible with some metric d.
5. X is a Fréchet space if X is a complete locally convex space with a metrizable topology
6. X is normable if a norm exists on X such that the metric induced by the norm is compatible with τ.

Definition 7. A **seminorm** on a vector space X is a real-valued function p on X such that:

1. $p(x + y) < p(x) + p(y)$ for all $x, y \in X$.

2. $p(\lambda x) = |\lambda| p(x)$ for all $x \in X$ and all scalars $\lambda \in \mathbb{K}$.

Definition 8. A seminorm p is a **norm** if it satisfies: $p(x) \neq 0$ if $x \neq 0$.

Definition 9. A family $\mathscr{P}$ of seminorms on X is said to be **separating** if to each $x \neq 0$ corresponds at least one $p \in \mathscr{P}$ with $p(x) \neq 0$.

Theorem 1. *With each separating family of seminorms on X we can associate a locally convex topology τ on X and vice versa: every locally convex topology is generated by some family of separating seminorms.*

Proof. c.f.: [9] □

Proof. c.f.: [9] □

Theorem 2. *A localy convex vector space (X, τ) is metrizable iff τ can be defined by $\mathscr{P} = \{p_n : n \in \mathbb{N}\}$ a countable separating family of seminorms on X. One can equip X with a metric which is compatible with τ and which provides a family of convex balls.*

Proof. c.f.: [10, 9] □

A lcv space from theorem 2 can be equipped with the metric:

$$d(x, y) := \sum_{\in \mathbb{N}} 2^n \frac{p_n(x-y)}{1 + p_n(x-y)} \tag{2}$$

This metric is compatible with τ but in general this metric doesn't provide convex balls (see the discussion in [9] after theorem 1.24 and exercise 18). If X is complete with respect to the metric from theorem 2 it is obviously a Fréchet space. Usually a Fréchet space topology is defined explicitely by gving a countable separating family of seminorms.

Definition 10. A **Banach space** is a normed tvs which is complete with respect to the norm.

Theorem 3. *A topological vector space X is normable if and only if its origin has a convex bounded neighborhood.*

Proof. c.f.: [9] □

2 Some historical remarks

First idea to generalize a notion of a manifold was a manifold modeled on a Banach space (Banach manifold). Later it turned out that Banach manifolds are not suitable for many questions of Global Analysis, as shown by the result due to [11], see also [12]: If a Banach Lie group acts effectively on a finite dimensional compact smooth manifold it must be finite dimensional itself. One would like to have a notion of a manifold modeled on a more general space: Fréchet or only locally convex.

Differential calculus in infinite dimensions has already quite a long history. Perhaps the need for such a generalization became apparent first to Bernoulli and Euler at the beginnings of variational calculus. During the 20-th century the motivation to differentiate in spaces which are more general than Banach spaces became stronger, and many different approaches and definitions were attempted. The main difficulty encountered was that composition of (continuous) linear mappings ceases to be a jointly continuous operation for any suitable topology on spaces of linear mappings.

Example 1 (after [2]). *Consider the evaluation $ev : E \times E^* \to \mathbb{R}$, where E is a locally convex space and E^* is its dual of continuous linear functionals equipped with any locally convex topology. Let us assume that the evaluation is jointly continuous. Then there are neighborhoods $U \subseteq E$ and $V \subseteq E^*$ of zero such that $ev(U \times V) \subseteq [-1, 1]$. But then U is contained in the polar of V , so it is bounded in E, and so E admits a bounded neighborhood and is thus normable.*

Definition 11. Given a dual pair (X, Y) the polar set or polar of a subset A of X is a set A° in Y defined as:

$$A^\circ := \{y \in Y : \sup\{|\langle x, y\rangle| : x \in A\} \leq 1\} \tag{3}$$

2.1 The notion of derivative

The problem of defining a derivative on a locally convex space roots in the variational calculus. Soon after the invention of the differential calculus, ideas were developed which would later lead to variational calculus. It evolved into a rather formal procedure, used extensively in physics. In his Lecture courses Weierstrass gave more reliable foundations to the theory, which was made public by Kneser (Kneser, A., *Lehrbuch der Variationsrechnung*, Vieweg, Braunschweig, 1900). Further development concerned mainly the relation between the calculus of variations and the theory of partial differential equations.

2.1.1 Fréchet derivative

Fréchet defined the derivative of a mapping f between normed spaces as follows:

Definition 12. Let $f : X \to Y$ be a mapping between two normed spaces. f is said to be Fréchet-differentiable if there exists a continuous linear operator A such that

$$\lim_{||h|| \to 0} \frac{f(x + h) - f(x) - Ah}{||h||} = 0. \tag{4}$$

2.1.2 Gâteaux derivative

Definition 13. Let X and Y be locally convex topological vector spaces, $U \subset X$ is open, and $F : X \to Y$. The Gâteaux differential $dF(u; h)$ of F at $u \in U$ in the direction $h \in X$ is defined as:

$$dF(u; h) = \lim_{\tau \to 0} \frac{F(u + \tau h) - F(u)}{\tau} = \frac{d}{d\tau} F(u + \tau h)\Big|_{\tau=0} \tag{5}$$

if the limit exists. If the limit exists for all $h \in X$, then one says that F is Gâteaux differentiable at u.

2.1.3 ... and more...

In [2] authors recall (after Averbukh, Smolyanov, *The various definitions of the derivative in linear topological spaces*, 1968) that in the literature one finds 25 inequivalent definitions of the first derivative (in tvs) in a single point. This shows that finite order differentiability beyond Banach spaces is really a nontrivial issue. For continuously differentiable mappings the many possible notions collapse to 9 inequivalent ones (fewer for Fréchet spaces). And if one looks for infinitely often differentiable mappings, then one ends up with 6 inequivalent notions (only 3 for Fréchet spaces).

2.2 Problem of cartesian closedness

One would like to have a property:

$$\mathcal{C}^{\infty}(E \times F, G) \cong \mathcal{C}^{\infty}(E, \mathcal{C}^{\infty}(F, G)) \qquad \text{(NOT TRUE!)}, \tag{6}$$

which is callad a "cartesian closednes". This is a property fulfilled by all the nice categories, but a category of smooth manifolds doesn't have it. This was a motivation for developing generalizations of smooth manifolds, so called smooth spaces:

- Chen spaces (Chen, 1977)
- diffeological spaces (Souriau 1980)

The categories of smooth spaces defined in those approaches have all the nice properties, but their objects are quite nasty.

There are also different approaches. In [5] (see also [2, 6]) a smooth calculus was proposed which has a property (6) holding without any restrictions for convenient vector spaces. The key idea is to define a different topology on the product. The approach of [5] is based on the bornological instead of topological concepts.

However if one wants to define a smooth manifold basing on a concept of charts, then the cartesian closedness is very limited even in the convenient setting (see discussion in [2], chapter IV). A way out is to base a definition of a manifold on the concept of the family of smooth mappings (see: [15, 16]).

2.3 Infinite dimensional Lie groups

In physics one would like to treat certain spaces of functions as infinite dimensional Lie groups. To put it in an appropriate mathematical setting one needs first a notion of an infinite dimensional manifold. A definition proposed in [1] makes it possible to provide certain infinite dimensional spaces with the structure of a manifold modeled on a locally convex vector space. One can apply in this case all tools of locally convex analysis. Unfortunately this definition doesn't cover all the interesting cases. In particular it fails for the spaces of mappings between noncompact manifolds.

Example 2 (after [1])**.** *If M is a non-compact finite-dimensional manifold, then one cannot expect the topological groups $\mathcal{C}^{\infty}(M, K)$ to be Lie groups. A typical example arises for $M = \mathbb{N}$ (a 0 - dimensional manifold) and $K = \mathbb{T} := \mathbb{R}/\mathbb{Z}$. Then $\mathcal{C}^{\infty}(M, K) \cong \mathbb{T}^{\mathbb{N}}$ is a topological group for which no 1-neighborhood is contractible.*

This means that one cannot consider $Diff(M)$ to be a ∞-dim Lie group for a noncompact manifold M. As globally hiperbolic lorentzian manifolds are noncompact, this result makes some of the physics applications impossible. An alternative approach is provided by [2]. This setting is more general but also has its drawbacks.

3 Proposed topics of the talks

1. Preliminaries to the convenient setting of global calculus (9 talks)

   - ***Bornological concepts in functional analysis I***
     Definition of bornology (family of bounded sets), relation between bornology and topology, bornologification, topologification, bornological vector spaces, linear bornology, examples of bornologies.
     Proposed literature: [7, 5], there is also a Diplomarbeit of Florian Gach, where all the definitions can be found: [13].
   - ***Bornological concepts in functional analysis II***
     Initial and final bornologies, duality with topology.
     Proposed literature: [7, 5, 13]
   - ***Mackey (bornological) convergence***
     Definition of Mackey convergence, Makey nets, Mackey-Cauchy sequences, completeness, comparison with other notions of convergence, examples.
     Proposed literature: [2, 7, 5, 13]
   - ***Mackey convergance in smooth calculus***
     Lipschitz curves, Mackey convergence of the difference quotioent.
     Proposed literature: [2] (chapter I, sections 1-2), [5] (section 2.2), [13](section 5), see also: [6].
   - ***The c^∞-topology***
     Definition of the c^∞-topology, definition of a convenient vector space, examples
     Proposed literature: [2] (chapter I, sections 2,4), [5], [6].
   - ***Cartesian closedness in convenient calculus I, II***
     Proofs of different variants of the exponential law in convenient setting.
     Proposed literature: [2] (chapter I, section 3), [5] (section 4.4), [6].
   - ***Uniform boundednes principle and it's consequences***
     Uniform boundednes principle, natural bornology vs. pointwise bornology, spaces of multilinear mappings.
     Proposed literature: [2] (chapter I, section 5), [5] (section 3.6), [6].
   - ***Importance of bornological concepts in smooth calculus***
     Summary of the previous talks.
     Proposed literature: [2, 5, 6].

2. Locally convex calculus (9 talks)

   (a) Introduction to the locally convex calculus (3 talks)

   - ***Locally convex vector spaces***
     Definition and important properties of locally convex vector spaces, Fréchet spaces
     Proposed literature: [9, 10, 1].
   - ***Calculus in locally convex vector spaces I, II***
     Definition of a derivative, Fundamental Theorem of Calculus, chain rule, partial derivatives, higher derivatives
     Proposed literature: [1, 3, 14].

(b) Infinite dimensional manifolds modeled on locally convex topological vector spaces. (2 talks)

- ***Infinite dimensional manifolds***
  Definition of an infinite dimensional manifold modeled on a locally convex tvs, properties, examples
  Proposed literature: [1, 14].
- ***Differential forms***
  Differential forms, exterior derivative, Lie derivative
  Proposed literature: [1, 14].

(c) Infinite dimensional Lie Groups (4 talks)

- ***Infinite dimensional Lie Groups***
  Definition of infinite dimensional Lie groups, Lie algebras
  Proposed literature: [1, 4].
- ***Groups of mappings***
  Examples of groups of mappings, diffeomorphism group of a compact manifold
  Proposed literature: [1, 4].
- ***Gauge group I, II***
  Gauge grup as an infinite dimensional Lie group, definition and properties
  Proposed literature: [1, 4].

3. Infinite dimensional manifolds in convenient setting
   Depending on the time left and interest of participants we can have a few talks on this subject at the end of the academic year.

4. Smooth spaces
   If there is someone interested in category theory, we can have also some talks on Chen spaces or diffeologies.

References

[1] Neeb, Karl-Hermann *Monastir Lecture Notes on Infinite-Dimensional Lie Groups*: http://www.math.uni-hamburg.de/home/wockel/data/monastir.pdf

[2] Kriegl, Andreas and Michor, Peter W. *The Convenient Setting of Global Analysis*: http://www.ams.org/online_bks/surv53/surv53.pdf

[3] Hamilton, R.S. *The Inverse Function Theorem of Nash and Moser*

[4] Wockel, Christoph *Infinite-Dimensional Lie Theory for Gauge Groups*: http://www.math.uni-hamburg.de/home/wockel/data/diss.pdf

[5] Frölicher, A.; Kriegl, A., *Linear spaces and differentiation theory*, Pure and Applied Mathematics, J. Wiley, Chichester, 1988.

[6] Frölicher, A.; *Axioms for convenient calculus*, Cahiers de topologie et géométrie différentielle catégoriques, tome **45**, no 4 (2004), p. 267-286

[7] Hogbe-Nlend, H.; *Bornologies and functional analysis*, Mathematics Studies no. 26, North-Holland, Amsterdam, New York, Oxford, 1977

[8] Bourbaki, N.; *Topological vector spaces*, ch. 1-5, Springer-Verlag Berlin, Heidelberg, New York 2003

[9] Rudin, W.; *Functional analysis*

[10] Köthe, G. ; *Topological vector spaces*

[11] Omori, H.; de la Harpe, P., *About interactions between Banach Lie groups and finite dimensional manifolds*, J. Math. Kyoto Univ. **12** (1972), 543-570

[12] Omori, H., *On Banach Lie groups acting on finite dimensional manifolds*, Tôhoku Math. J. **30** (1978), 223-250

[13] Gach, F.; *Topological versus Bornological Concepts in Infinite Dimensions*, Diplomarbeit zur Erlangung des akademischen Grades Magister rerum naturalium, 2004 Wien, betreut von Ao. Univ. Prof. Dr. Andreas Kriegl

[14] Glöckner, H; *Infinite-dimensional Lie groups without completeness condition*, Geometry and Analysis on finite and infinite-dimensional Lie groups, Eds. A. Strasburger, W. Wojtynski, J. Hilgert and K.-H. Neeb, Banach Center Publications **55** (2002), 43-59.

[15] Seip, U.; *A convenient setting for smooth manifolds*, J. Pure Appl. Algebra **21** (1981), 279-305.

[16] Kriegl, A.; *Eine Theorie glatter Mannigfaltigkeiten und Vektorbündel*, Dissertation, Universität Wien, 1980.